DAHLGREN DIVISION
NAVAL SURFACE WARFARE CENTER
Panama City, Florida 32407-7001

CSS/TR-02/10

PERSONAL COMPUTER SHALLOW WATER ACOUSTIC TOOL-SET (PC SWAT) 7.0: LOW FREQUENCY PROPAGATION AND SCATTERING

GARY STEVEN SAMMELMANN

LITTORAL WARFARE TECHNOLOGY AND SYSTEMS DEPARTMENT

COASTAL SYSTEMS STATION

JUNE 2002

REPORT DOCUMENTATION PAGE

Form Approved
OBM No. 0704-0188

1. AGENCY USE ONLY (*Leave blank*)	2. REPORT DATE	3. REPORT TYPE AND DATES COVERED
	JUNE 2002	

4. TITLE AND SUBTITLE

PERSONAL COMPUTER SHALLOW WATER ACOUSTIC TOOL-SET (PC SWAT) 7.0: LOW FREQUENCY PROPAGATION AND SCATTERING

5. FUNDING NUMBERS

6. AUTHOR(S)

Gary Steven Sammelmann

7. PERFORMING ORGANIZATION NAME(S) AND ADDRESS(ES)

ATTN: R21
Commanding Officer
CSS DD NSWC
6703 West Highway 98
Panama City, FL 32407-7001

8. PERFORMING ORGANIZATION REPORT NUMBER

CSS/TR-02/10

9. SPONSORING/MONITORING AGENCY NAME(S) AND ADDRESS(ES)

Chief of Naval Research
800North Quincy Street
Arlington, Virginia 22217-5000

10. SPONSORING/MONITORING AGENCY REPORT NUMBER

11. SUPPLEMENTARY NOTES

12a. DISTRIBUTION/AVAILABILITY STATEMENT

Approved for public release; distribution is unlimited

12b. DISTRIBUTION CODE

A

13. ABSTRACT (*Maximum 200 words*)

This report describes the theoretical basis behind the low frequency propagation and scattering models used in Personal Computer Shallow Water Acoustics Tool-set (PC SWAT 7.0). PC SWAT is a user-friendly sonar simulation developed by Dr. Sammelmann. It is used widely throughout the Department of Defense.

14. SUBJECT TERMS

PERSONAL COMPUTER SHALLOW WATER ACOUSTIC TOOL-SET (PC SWAT 7.0), SONAR, MINEHUNTING, MINE COUNTERMEASURES

15. NUMBER OF PAGES

162

16. PRICE CODE

17. SECURITY CLASSIFICATION OF REPORT	18. SECURITY CLASSIFICATION OF THIS PAGE	19. SECURITY CLASSIFICATION OF ABSTRACT	20. LIMITATION OF ABSTRACT
UNCLASSIFIED	UNCLASSIFIED	UNCLASSIFIED	SAR

NSN 7540-01-280-5500

Standard Form 298 (Rev 2-89)

FOREWORD

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This report has been reviewed and approved by the Littoral Warfare Technology and Systems Department and Delbert C. Summey.

Approved by

Delbert C. Summey, Head
Littoral Warfare Technology and
Systems Department

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1.0 INTRODUCTION

This report describes the theoretical basis for the low frequency propagation and scattering models used by PC SWAT 7.0. These models are based on the normal mode description of propagation of sound in a range independent waveguide.

In addition to the Introduction and Reference sections, this report is composed of thirty sections. Section 2 gives a brief derivation of the acoustic wave equation. Section 3 provides a brief review of the theory of Sturm-Louville operators. Sections 4, 5, and 6 provide a description of the construction of the Greens' Function in a horizontally stratified waveguide. Section 7 provides a derivation of the normalization of the depth functions in a waveguide in terms of the boundary conditions at the top and bottom of the waveguide. Section 8 describes the construction of the normal mode description of propagation in a waveguide with a piece-wise constant sound speed and density profile and a rigid basement. Section 9 describes the extension of the methods of Section 8 to include a homogeneous half-space as the basement of the waveguide. Section 10 extends the normal mode description to include piece-wise linear sound speed profiles. Sections 11 and 12 provide examples of the normal mode description in an isovelocity waveguide with a rigid and homogeneous half-space basement. Sections 13 and 14 describe the effects of small-scale roughness of the coherent (mean) field. Section 15 provides the derivation of the time domain description of the propagation of normal modes based on a saddle point analysis of the propagation of a normal mode. Sections 16, 17, and 18 describe representations of cylindrical and spherical wave functions, and the transformations between these two sets of solutions of the free-field Helmholtz Equation. Section 19 describes the expansion of the normal mode contributions to propagation in terms of a spherical basis set about the source and receive points. Section 20 describes the use of the multipole expansion of the Greens' Function in the inclusion of the effects of the directivity of the source and the receiver on the acoustic field. Section 21 describes the effects of rotating the local coordinate system on the spherical expansions of the Greens' Function. Section 22 presents a review of the Helmholtz Integral Equation in preparation for the introduction of the spherical T-matrix approach to scattering from a target. Sections 23 and 24 derive the spherical T-matrix approach for scattering from rigid and elastic solids. Section 25 describes the scattering from an elastic target in a waveguide based on the preceding section on the spherical T-matrix. Sections 26, 27, and 28 describe the normal mode and time domain representation of the scattering from a rough interface and volume inhomogeneities. Section 29 describes the plane wave approximation of the computation of the signal-to-noise ratio using the normal mode method. Section 30 describes the global matrix method of solving for the normal modes, and Section 31 presents some examples computed using PC SWAT 7.0.

2.0 ACOUSTIC WAVE EQUATION

Conservation of mass and momentum are given by the hydrodynamic equations of motion[1,2]:

$$\frac{\partial}{\partial t}\rho + \vec{\nabla} \bullet (\rho\vec{v}\,) = 0 \tag{2.1}$$

$$\rho\frac{\partial}{\partial t}\vec{v} + \rho\vec{v} \bullet \vec{\nabla} v = -\vec{\nabla} P + \rho\vec{F} \tag{2.2}$$

where ρ is the density of the fluid, $\vec{v}$ is the velocity of the fluid, P is the pressure of the fluid, and F is the external force per unit mass (for example, gravity). The acoustic wave equation is defined by linearizing the above hydrodynamic equations by making the following change of variables in the above equations.

$$\rho = \rho_0 + \delta\rho \tag{2.3}$$

$$\vec{v} = \vec{v}_0 + \delta\vec{v} \tag{2.4}$$

$$P = P_0 + p \tag{2.5}$$

The linearized equations of motion are:

$$\frac{\partial}{\partial t}\delta\rho + \vec{\nabla} \bullet (\delta\rho\vec{v}_0) + \vec{\nabla} \bullet (\rho_0\delta\vec{v}) = 0 \tag{2.6}$$

$$\frac{\partial}{\partial t}\delta\vec{v} + \vec{v}_0 \bullet \vec{\nabla}\delta\vec{v} + \delta\vec{v} \bullet \vec{\nabla}\vec{v}_0 = -\frac{1}{\rho_0}\vec{\nabla} p + \frac{\delta\rho}{\rho_0^2}\vec{\nabla} P_0 \tag{2.7}$$

If one neglects the thermal conductivity of the fluid, and diffusion of the components of the fluid, the acoustic propagation of a wave can be treated as an adiabatic process, where the density and pressure fluctuations are proportional:

$$\delta\rho = c^{-2} p \tag{2.8}$$

In the above equation, c is the speed of sound in the fluid given by the partial derivative of the pressure in the fluid with respect to density at constant entropy:

$$c^2 = (\frac{\partial p}{\partial \rho})(\delta S = 0) \tag{2.9}$$

Assume that in the absence of the wave the medium is at rest. In addition, neglect gravitational effects on the wave, that is, the gradient of the background pressure field in Equation 2.7. In this case, the linearized equations of motion are:

$$\rho_0 \frac{\partial \delta \vec{v}}{\partial t} = -\vec{\nabla} p \tag{2.10}$$

$$\frac{\partial p}{\partial t} = \rho_0 c^2 \vec{\nabla} \bullet \delta \vec{v} \tag{2.11}$$

Eliminating the velocity of the acoustic wave in the above pair of first order differential equations, one arrives at the following second order differential equation in terms of the pressure of the acoustic wave for a stationary medium:

$$\vec{\nabla} \frac{1}{\rho_0} \bullet \vec{\nabla} p + \frac{\partial}{\partial t}\left(\frac{1}{\rho_0 c^2} \frac{\partial p}{\partial t}\right) = 0 \tag{2.12}$$

In the case of a monochromatic wave with $e^{-i\omega t}$ time dependence, the acoustic wave equation takes on the following form:

$$\rho_0 \vec{\nabla} \bullet \frac{1}{\rho_0} \vec{\nabla} p + k^2 p = 0 \tag{2.13}$$

where

$$k = \frac{\omega}{c} \tag{2.14}$$

is the acoustic wavenumber of the wave.

3. STURM-LOUVILLE OPERATORS

This section provides a brief overview of the theory of Sturm-Louville operators. An overview of the Sturm-Louville Equation can be found in Morse and Feshbach[3,4].

The Sturm-Louville Differential Equation is a linear, second order differential equation of the following form:

$$-\frac{d}{dx}p(x)\frac{d}{dx}\psi(x)+(q(x)-\lambda w(x))\psi(x)=0 \tag{3.1}$$

The differential operator

$$L=\frac{1}{w(x)}\{-\frac{d}{dx}p(x)\frac{d}{dx}+(q(x)-\lambda w(x))\} \tag{3.2}$$

is a self-adjoint operator on the space of smooth functions $C^{\infty}(a,b)$ on which the following inner product is defined:

$$<\psi\,|\,\psi'>=\int_a^b \psi(x)\psi'(x)w(x)dx$$

THEOREM 1

This theorem establishes the criterion for a pair of solutions of the Sturm-Louville Equation to be linearly independent.

If the functions *p(x)* and *q(x)* are continuous function in the interval (a,b) and $\psi_1(x)$ and $\psi_2(x)$ are a pair of solutions of the differential equation:

$$\frac{d^2}{dx^2}\psi(x)+p(x)\frac{d}{dx}\psi(x)+q(x)\psi(x)=0 \tag{3.3}$$

for which the Wronskian *(W)* is defined below:

$$W(\psi_1,\psi_2)=\psi_1(x)\frac{d\psi_2(x)}{dx}-\frac{d\psi_1(x)}{dx}\psi_2(x) \tag{3.4}$$

is non-zero for a point in the interval (a,b). Then every solution of this differential equation in this interval is a linear combination of these two functions.

THEOREM 2

This theorem yields information about the Wronskian of a pair of solutions of the Sturm-Louville Equation.

If $\psi_1(x)$ and $\psi_2(x)$ are solutions of the Sturm-Louville Differential Equation:

$$-\frac{d}{dx}p(x)\frac{d}{dx}\psi(x)+(q(x)-\lambda w(x))\psi(x)=0 \tag{3.5}$$

Then the Wronskian of these two functions is of the following form:

$$W(\psi_1,\psi_2)=\psi_1(x)\frac{d\psi_2(x)}{dx}-\frac{d\psi_1(x)}{dx}\psi_2(x)=Const/p(x) \tag{3.6}$$

THEOREM 3

This theorem states that two eigenfunctions of the Sturm-Louville Equation are necessarily orthogonal if they have different eigenvalues. Note, the converse statement is not true.

Let $\psi(x)$ and $\psi'(x)$ be two eigenfunctions of the Sturm-Louville Equation:

$$-\frac{d}{dx}p(x)\frac{d}{dx}\psi(x)+(q(x)-\lambda w(x))\psi(x)=0 \tag{3.7}$$

with eigenvalues $\lambda \neq \lambda'$, respectively. Then the inner product

$$<\psi\,|\,\psi'>=\int_a^b \psi(x)\psi'(x)w(x)dx=0$$

vanishes.

THEOREM 4

This theorem describes the orthonormality and completeness of the spectrum of the Sturm-Louville Equation. Equations 3.9 and 3.10 describe the orthonormality of a suitable basis of eigenfunctions of the Sturm-Louville Equation. Equation 3.11 describes the completeness condition for this basis.

Suppose the eigenvectors of the Sturm-Louville Equation

$$-\frac{d}{dx}p(x)\frac{d}{dx}\psi(x)+(q(x)-\lambda w(x))\psi(x)=0 \tag{3.8}$$

consist of a discrete set of eigenvectors $\psi_m(x)$ with a discrete spectrum and a continuous set of eigenvectors $\psi_\lambda(x)$ with a continuous spectrum. Then one can normalize a complete set of eigenvectors, such that the following orthonormality conditions are valid:

$$<\psi_m \mid \psi_n>=\delta_{m,n} \tag{3.9}$$

$$<\psi_\lambda \mid \psi_{\lambda'}>=\delta(\lambda-\lambda') \tag{3.10}$$

and the following completeness condition is valid.

$$\sum_m \psi_m(x)\psi_m(x')+\frac{1}{2\pi i}\int \psi_\lambda(x)\psi_\lambda(x')d\lambda=\frac{1}{w(x)}\delta(x-x') \tag{3.11}$$

THEOREM 5

This theorem describes the construction of the Green's Function for the Sturm-Louville Equation from a complete orthonormal basis of the spectrum of the equation.

Let $G(x,x':\lambda)$ be a solution of the inhomogeneous Sturm-Louville Equation:

$$-\frac{d}{dx}p(x)\frac{d}{dx}G(x,x':\lambda)+(q(x)-\lambda w(x))G(x,x':\lambda)=-\delta(x-x') \tag{3.12}$$

subject to the boundary conditions:

$$\alpha_1 G(a,x':\lambda)+\beta_1\frac{d}{dx}G(a,x':\lambda)=0 \tag{3.13}$$

$$\alpha_2 G(b,x':\lambda)+\beta_2\frac{d}{dx}G(b,x':\lambda)=0 \tag{3.14}$$

Let $\psi_m(x)$ be a complete, orthonormal basis of the discrete spectrum, and $\psi(x,\lambda)$ be a complete, orthonormal basis of the continuous spectrum as described in Theorem 4. Then the solution of the above differential equation for the Green's Function of the Sturm-Louville Equation is of the form:

$$G(x,x':\lambda)=\sum\frac{\psi_m(x)\psi_m(x')}{(\lambda-\lambda_m)}+\frac{1}{2\pi i}\int\frac{\psi(x,\lambda')\psi(x',\lambda')}{(\lambda-\lambda')} \tag{3.15}$$

in terms of this basis.

THEOREM 6:

This theorem describes an alternative solution for the Green's Function of the Sturm-Louville Equation, which does not require the construction of an orthonormal basis of the homogeneous equation.

Let $G(x,x':\lambda)$ be a solution of the inhomogeneous Sturm-Louville Equation:

$$-\frac{d}{dx}p(x)\frac{d}{dx}G(x,x':\lambda)+(q(x)-\lambda w(x))G(x,x':\lambda)=-\delta(x-x') \tag{3.16}$$

subject to the boundary conditions.

$$\alpha_1 G(a,x':\lambda)+\beta_1\frac{d}{dx}G(a,x':\lambda)=0 \tag{3.17}$$

$$\alpha_2 G(b,x':\lambda)+\beta_2\frac{d}{dx}G(b,x':\lambda)=0 \tag{3.18}$$

Let $\psi_1(x)$ and $\psi_2(x)$ be a pair of solutions of the differential equation:

$$-\frac{d}{dx}p(x)\frac{d}{dx}\psi_i(x)+(q(x)-\lambda w(x))\psi_i(x)=0 \tag{3.19}$$

and the boundary condition:

$$\alpha_1\psi_1(a)+\beta_1\frac{d}{dx}\psi_1(a)=0 \tag{3.20}$$

$$\alpha_{21}\psi_2(b)+\beta_2\frac{d}{dx}\psi_2(b)=0 \tag{3.21}$$

at x = a, and x = b, respectively. Then the Green's Function can be represented in the form:

$$G(x,x'\!:\lambda) = -\frac{\psi_1(x_<)\psi_2(x_>)}{W(x')} \tag{3.22}$$

in terms of the functions $\psi_1(x)$ and $\psi_2(x)$, where

$$W(x) = \psi_1(x)\frac{d\psi_2(x)}{dx} - \frac{d\psi_1(x)}{dx}\psi_2(x) \tag{3.23}$$

is the Wronskian of these two functions.

4. GREEN'S FUNCTION FOR A RANGE INDEPENDENT OCEAN

This section gives a brief overview of the construction of the Green's Function for a range independent waveguide. One must adopt a cylindrical coordinate system where the upper surface is given by the plane z =0, and the z-axis is oriented downward. The source is located at a
depth z_s at a horizontal range r = 0.

The Green's Function for the acoustic wave equation in a constant density ocean is a solution of the differential equation:

$$\left(\frac{d^2}{dr^2}+\frac{1}{r}\frac{d}{dr}+\frac{1}{r^2}\frac{d^2}{d\phi^2}+\frac{d^2}{dz^2}+k^2\right)G(\vec{r},\vec{r}_s)=-\frac{1}{r}\delta(r-r_s)\delta(z-z_s)\delta(\phi-\phi_s) \tag{4.1}$$

subject to appropriate boundary conditions.

Make the simplifying assumption that the speed of sound is a function of the depth coordinate only. Make the further simplifying assumption that the source is omni-directional and located a horizontal range r = 0. In this case, Green's Function simplifies to the following differential equation:

$$\left(\frac{d^2}{dr^2}+\frac{1}{r}\frac{d}{dr}+\frac{d^2}{dz^2}+k(z)^2\right)G(r,z,z_s)=-\frac{1}{2\pi r}\delta(r)\delta(z-z_s) \tag{4.2}$$

Here

$$G(r,z,z_s)=\frac{1}{2\pi}\int_0^{2\pi}G(r,r_s=0,\phi,\phi_s,z,z_s)\,d\phi \tag{4.3}$$

is the relationship of the three-dimensional Green's Function of Equation 4.1 to the two-dimensional Green's Function of Equation 4.2.

Neglecting the inhomogeneous term on the right hand side of Equation 4.2, the left hand side of the differential equation is separable. Define the one-dimensional Green's Functions for the radial and depth coordinate as solutions of the following differential equations for a given separation parameter λ:

$$\frac{d^2}{dr^2}G_r(r:\lambda)+\frac{1}{r}\frac{d}{dr}G_r(r:\lambda)+\lambda G_r(r:\lambda)=-\frac{1}{2\pi r}\delta(r) \tag{4.4}$$

$$\frac{d^2}{dz^2} G_z(z,z_s : \lambda) + (k(z)^2 - \lambda) G_z(z,z_s : \lambda) = -\delta(z - z_s) \tag{4.5}$$

The two-dimensional Green's Function $G(r,z,z_s)$ can be written as the spectral integral

$$G(r,z,z_s) = \frac{1}{2\pi i} \int_C G_r(r : \lambda) G_z(z,z_s : \lambda) d\lambda \tag{4.6}$$

of the radial and depth Green's Function over an appropriate Contour C. The Contour C can be either chosen to enclose all the poles of the radial Green's Function or the depth Green's Function in a positive sense, that is, the Contour C is chosen such that either the spectral integral

$$\frac{1}{2\pi i} \int_C G_r(r : \lambda) d\lambda = \frac{1}{2\pi r} \delta(r) \tag{4.7}$$

over the radial Green's Function is equal to the Dirac Delta Function with respect to range, or the spectral integral

$$\frac{1}{2\pi i} \int_C G_z(z,z_s : \lambda) d\lambda = \delta(z - z_s) \tag{4.8}$$

over the depth Green's Function is equal to the Dirac Delta Function with respect to depth. The fact that such a contour may be chosen is due to the fact that both the radial and depth dependent Green's Functions are the Green's Functions of a Sturm-Louville Differential Equation.

Now one may make the change of variables:

$$q^2 = \lambda \tag{4.9}$$

One can express the spectral integral for the Green's Function as the integral:

$$G(r,z,z_s) = \frac{1}{\pi i} \int_{-\infty + i0}^{+\infty - i0} G_r(r : q) G_z(z,z_s : q) q dq \tag{4.10}$$

5. RADIAL GREEN'S FUNCTION

This section presents a derivation of the radial Green's Function.

The radial Green's Function for a range independent waveguide satisfies the following differential equation,

$$\frac{d^2}{dr^2}G_r(r:q)+\frac{1}{r}\frac{d}{dr}G_r(r:q)+q^2G_r(r:q)=-\frac{1}{2\pi r}\delta(r) \tag{5.1}$$

subject to the boundary conditions

$$\lim_{r\to 0}[r\frac{d}{dr}G_r(r)]=-\frac{1}{2\pi}$$

$$\lim_{r\to\infty}\sqrt{r}(\frac{d}{dr}G_r(r)-iqG_r(r))=0$$

at the origin and at infinity. The boundary condition at the origin follows from integrating Equation 5.1 in an infinitesimal region about the origin. The boundary condition at infinity follows from the requirement that the Green's Function asymptotically approaches an outgoing plane wave.

Equation 5.1 is the Bessel Differential Equation of Order 0. Thus, the Green's Function is of the form:

$$G_r(r:q)=AH_0^{(1)}(qr)+BH_0^{(2)}(qr)$$

where $H_0^{(1)}(qr)$ and $H_0^{(2)}(qr)$ are the Hankel Functions of the first and second kind. The boundary condition at infinity allows us to discard the term with the Hankel Function of the second kind. The boundary condition at the origin determines the coefficient A. The radial Green's Function is given by the following expression:

$$G_r(r:q)=\frac{i}{4}H_0^{(1)}(qr)$$

6. CONSTRUCTION OF THE DEPTH-DEPENDENT GREEN'S FUNCTION

This section describes the construction of the depth dependent Green's Function from solutions of the homogeneous differential equation.

The depth dependent Green's Function is a solution of the following differential equation.

$$LG_2(z=0,z_s;q) \equiv \rho(z)\frac{\partial}{\partial z}\frac{1}{\rho(z)}\frac{\partial G_2(z,z_s;q)}{\partial z}+(k^2(z)-q^2)G_2(z,z_s;q)=-\delta(z-z_s) \tag{6.1}$$

The Green's Function is subject to the following boundary conditions.

$$B_1G_2(z=0,z_s;q) \equiv f^T(q)G_2(z=0,z_s;q)+\frac{g^T(q)}{\rho(0)}\frac{\partial G_2(z=0,z_s;q)}{\partial z}=0 \tag{6.2}$$

$$B_2G_2(z=d,z_s;q) \equiv f^B(q)G_2(z=d,z_s;q)+\frac{g^B(q)}{\rho(d)}\frac{\partial G_2(z=d,z_s;q)}{\partial z}=0 \tag{6.3}$$

The Green's Function satisfies the following conditions in the limit z approaches the source depth:

$$\lim_{\varepsilon\to 0}(G_2(z_s+\varepsilon,z_s)-G_2(z_s-\varepsilon,z_s))=0 \tag{6.4}$$

$$\lim_{\varepsilon\to 0}(\frac{\partial G_2(z_s+\varepsilon,z_s)}{\partial z}-\frac{\partial G_2(z_s-\varepsilon,z_s)}{\partial z})=-\rho(z_s) \tag{6.5}$$

Let $p_1(z)$ and $p_2(z)$ be a pair of solutions of the homogeneous differential equation that satisfy the boundary condition at the surface and bottom respectively, that is, they satisfy the differential equation

$$Lp_1(z)=0 \tag{6.6}$$

$$Lp_2(z)=0 \tag{6.7}$$

and boundary condition

$$B_1p_1(z=0)=0 \tag{6.8}$$

$$B_2p_2(z=d)=0 \tag{6.9}$$

The depth dependent Green's Function is given by the following expression:

$$G_2(z,z_s;q) = -\frac{p_1(z_<)p_2(z_>)}{W(z_s;q)} \tag{6.10}$$

where

$$W(z;q) = p_1(z)\frac{\partial p_2(z)}{\partial z} - \frac{\partial p_1(z)}{\partial z}p_2(z) \tag{6.11}$$

is the Wronskian of the two solutions, and

$$z_< = \min(z,z_s) \tag{6.12}$$

$$z_> = \max(z,z_s)\,. \tag{6.13}$$

7. NORMALIZATION OF DEPTH FUNCTIONS

This section describes the normalization condition of the depth functions and the depth dependent Green's Function following the discussion by Porter[4].

The depth function is a solution of the following differential equation:

$$L(q)\psi(z) \equiv \rho(z)\frac{\partial}{\partial z}\left(\frac{1}{\rho(z)}\frac{\partial \psi(z)}{\partial z}\right) + (k^2(z) - q^2)\psi(z) = 0 \tag{7.1}$$

and the following boundary conditions:

$$B_1(q)\psi(z=0) \equiv f^T(q)\psi(z=0) + \frac{g^T(q)}{\rho(0)}\frac{\partial \psi(z=0)}{\partial z} = 0 \tag{7.2}$$

$$B_2(q)\psi(z=d) \equiv f^B(q)\psi(z=d) + \frac{g^B(q)}{\rho(d)}\frac{\partial \psi(z=d)}{\partial z} = 0 \tag{7.3}$$

In general, a non-zero solution of the above differential equation and boundary conditions only exists for a discrete spectrum of eigenvalues.

Let $p_1(z)$ and $p_2(z)$ be a pair of unforced solutions of the differential equation:

$$L(q)p_1(z) = 0 \tag{7.4}$$

$$L(q)p_2(z) = 0 \tag{7.5}$$

and boundary condition at the surface (bottom), respectively:

$$B_1(q)p_1(z=0) = 0 \tag{7.6}$$

$$B_2(q)p_2(z=d) = 0 \tag{7.7}$$

Here, boundary conditions are only imposed on $p_1(z)$ and $p_2(z)$ at one end of the domain, respectively. This relaxation of the boundary condition at one end of the domain allows the existence of a continuous spectrum of solutions. The functions $p_1(z)$ and $p_2(z)$ are analogous to the left-going and right-going traveling waves on a one-dimensional string. The depth functions are analogous to the standing waves on this string when both ends of the string are clamped.

Let

$$W(z,q) = p_1(z)\frac{\partial p_2(z)}{\partial z} - \frac{\partial p_1(z)}{\partial z}p_2(z) \tag{7.8}$$

be the Wronskian of the above pair of solutions of this differential equation. The two solutions are said to be linearly independent if the Wronskian is non-zero.

Recall the depth dependent Green's Function may be represented in the form

$$G_2(z,z_s;q) = -\frac{p_1(z_<)p_2(z_>)}{W(z_s;q)} \tag{7.9}$$

in terms of this pair of solutions and the Wronskian. The spectrum of normal modes is comprised of those values of q for which the above Wronskian vanishes:

$$W(z_s;q) = 0 \tag{7.10}$$

Let $\psi(z)$ be the depth function at eigenvalue q_m at which the above Wronskian vanishes. Without loss of generality, one may assume $p_1(z)$ and $p_2(z)$ form a one-parameter family of solutions of the depth equation that satisfy the limiting conditions:

$$\lim_{q\to q_m} p_1(z) = \psi(z) \tag{7.11}$$

$$\lim_{q\to q_m} p_2(z) = \psi(z) \tag{7.12}$$

that they approach the depth function $\psi(z)$ as q approaches the eigenvalue q_m.

The following relationship:

$$p_2(z)L(q_m)\psi(z) - (L(q)p_2(z))\psi(z) =$$

$$\frac{\partial}{\partial z}((p_2(z)\frac{\partial \psi(z)}{\partial z} - \frac{\partial p_2(z)}{\partial z}\psi(z))/\rho(z)) + (q^2 - q_m^2)\frac{p_2(z)\psi(z)}{\rho(z)} = 0 \tag{7.13}$$

between $\psi(z)$ and $p_2(z)$ follows from the fact these two functions are solutions of the depth equation for eigenvalue q_m and q, respectively. Integrating the above quantity over the water column, one arrives at the expression

$$((p_2(z)\frac{\partial \psi(z)}{\partial z} - \frac{\partial p_2(z)}{\partial z}\psi(z))/\rho(z))\Big|_{z=0}^{D} = -(q^2 - q_m^2)\int_0^D p_2(z)\psi(z)/\rho(z)dz \tag{7.14}$$

Similarly, the relationship:

$$p_1(z)L(q)p_2(z)-(L(q)p_1(z))p_2(z)=\frac{\partial}{\partial z}((p_1(z)\frac{\partial p_2(z)}{\partial z}-\frac{\partial p_1(z)}{\partial z}p_2(z))/\rho(z))=0 \tag{7.15}$$

proves that the ratio

$$W(z;q)/\rho(z)=(p_1(z)\frac{\partial p_2(z)}{\partial z}-\frac{\partial p_1(z)}{\partial z}p_2(z))/\rho(z) \tag{7.16}$$

of the Wronskian and the density is a constant.

Using the relationship:

$$p_2(z)\frac{\partial \psi(z)}{\partial z}-\frac{\partial p_2(z)}{\partial z}\psi(z)=(\frac{1}{\psi(z)}\frac{\partial \psi(z)}{\partial z}-\frac{1}{p_2(z)}\frac{\partial p_2(z)}{\partial z})p_2(z)\psi(z) \tag{7.17}$$

and the boundary conditions

$$\frac{1}{\psi(z=D)}\frac{\partial \psi(z=D)}{\partial z}=-\frac{f^B(q_m)}{g^B(q_m)}\rho(D) \tag{7.18}$$

one arrives at the expression

$$p_2(D)\frac{\partial \psi(D)}{\partial z}-\frac{\partial p_2(D)}{\partial z}\psi(D)=(\frac{f^B(q)}{g^B(q)}-\frac{f^B(q_m)}{g^B(q_m)})\rho(D)p_2(D)\psi(D) \tag{7.19}$$

Since the depth function $\psi(z)$ satisfies the boundary condition at both the top and bottom surface, one has the following expression for the impedance of the depth function at the top surface:

$$\frac{1}{\psi(0)}\frac{\partial \psi(0)}{\partial z}=-\frac{f^T(q_m)}{g^T(q_m)}\rho(0) \tag{7.20}$$

Utilizing the fact that the ratio of the Wronskian and density is a constant one obtains the following relationship for the impedance of the function $p_2(z)$ at the top surface:

$$\frac{1}{p_2(0)}\frac{\partial p_2(0)}{\partial z}=\frac{1}{p_1(0)}\frac{\partial p_1(0)}{\partial z}+\frac{W(0;q)}{\rho(0)}=-\frac{f^T(q)}{g^T(q)}\rho(0)+\frac{W(z:q)}{\rho(z)p_1(0)p_2(0)} \tag{7.21}$$

Using Equations 7.17, 7.20, and 7.21 one arrives at the following expression:

$$p_2(0)\frac{\partial \psi(0)}{\partial z}-\frac{\partial p_2(0)}{\partial z}\psi(0)$$

$$=(\frac{f^T(q)}{g^T(q)}-\frac{f^T(q_m)}{g^T(q_m)})\rho(0)p_2(0)\psi(0)+(W(z:q)-W(z:q_m)\psi(0)/p_1(0) \quad (7.22)$$

Here, use of the vanishing of the Wronskian

$$W(z:q_m)=0 \quad (7.23)$$

at the eigenvalue q_m has been made to add a term that is proportional to this term to Equation 7.22.

Substituting Equations 7.19 and 7.21 into Equation 7.14, one arrives at the following expression for the normalization of the depth functions:

$$(q^2-q_m^2)\int_0^D\frac{p_2(z)\psi(z)}{\rho(z)}dz+(\frac{f^B(q)}{g^B(q)}-\frac{f^B(q_m)}{g^B(q_m)})p_2(D)\psi(D)+$$

$$-(\frac{f^T(q)}{g^T(q)}-\frac{f^T(q_m)}{g^T(q_m)})p_2(0)\psi(0)-\frac{(W(z:q)-W(z:q_m)}{\rho(z)}\psi(0)/p_1(0)=0 \quad (7.24)$$

Divide this equation by $(q^2-q_m^2)$ and take the limit q approaches q_m to obtain the following relationship between the Wronskian and the normalization of the depth functions.

$$\frac{1}{\rho(z)}\frac{\partial W(z,q_m)}{\partial z}=2q_m\int_0^D\frac{\psi(z)\psi(z)}{\rho(z)}dz-\frac{\partial}{\partial q}(\frac{f^T(q_m)}{g^T(q_m)})\psi(0)^2+\frac{\partial}{\partial q}(\frac{f^B(q_m)}{g^B(q_m)})\psi(D)^2 \quad (7.25)$$

Without loss of generality, one can normalize the depth functions such that the partial derivative of the Wronskian at the m'th normal mode is given by the expression:

$$\frac{\partial W(z;q_m)}{\partial q}=2q_m\rho(z) \quad (7.26)$$

by requiring the following normalization condition on the depth function:

$$\int_0^D\frac{\psi(z)\psi(z)}{\rho(z)}dz-\frac{1}{2q_m}\frac{\partial}{\partial q}(\frac{f^T(q_m)}{g^T(q_m)})\psi(0)^2+\frac{1}{2q_m}\frac{\partial}{\partial q}(\frac{f^B(q_m)}{g^B(q_m)})\psi(D)^2=1 \quad (7.27)$$

In the case of a pressure release surface and a rigid bottom, the partial derivative terms in the above expression vanishes and the previous expression reduces to the usual normalization condition in the case the spectrum is discrete.

8. N HOMOGENEOUS FLUID LAYERS OVER A RIGID BOTTOM

This section describes the construction of the Green's Function for N fluid layers over a rigid bottom.

The Green's Function is given by the following spectral integral:

$$G(r,z,z_s)=\frac{1}{4\pi}\int_{-\infty+i0}^{+\infty-i0} H_0^{(1)}(qr)G_z(z,z_s:q)qdq \tag{8.1}$$

where $G_z(z,z_s:q)$ is the depth dependent Green's Function. This spectral integral can be represented as a sum of residues of poles of the depth dependent Green's Function in the upper half q-plane. The remainder of this section will be devoted to solving the characteristic equation for the location of these poles, and the evaluation of the depth functions.

Let $\{\rho_n, c_n : n = 0,1..N-1\}$ denote the density and sound speed in the N layers. Let $\{z_n : n = 0,1...N\}$ denote the z-coordinate of the N+1 interfaces. Without loss of generality, one may assume $z_0 = 0$, and $z_N = D$.

The depth function in the n'th layer is required to satisfy the differential equation:

$$\frac{dF(z)}{dz}+(k_n^2-q^2)=0 \tag{8.2}$$

where $k_n = \omega / c_n$ is the wavenumber in the n'th layer. The functions:

$$F_n^+(z)=e^{+ih_n(z-z_n)} \tag{8.3}$$

$$F_n^-(z)=e^{-ih_n(z-z_n)} \tag{8.4}$$

denote a basis for the downward and upward going waved in the n'th layer, where

$$h_n(q)=i\sqrt{q^2-k_n^2} \tag{8.5}$$

is the vertical wavenumber in the n'th layer, and q is the horizontal wavenumber. Here one adopts the conventions of Ewing, Jardetsky, and Press[6] for the vertical wavenumber; that is, the imaginary component of the vertical wavenumber is greater than zero on the physical sheet, and the branch cut is the hyperbola on which the imaginary component is zero in the complex q-plane.

Let the following sum represent the depth function in the n'th layer:

$$F_n(z) = A_n^+ F_n^+(z) + A_n^- F_n^-(z) \tag{8.6}$$

of upward and downward-going waves. The depth function is required to satisfy the following boundary conditions.

$$F_0(z_0) = 0 \tag{8.7}$$

$$F_n(z_{n+1}) = F_{n+1}(z_{n+1}),\ n = 0,1..(N-2) \tag{8.8}$$

$$\frac{1}{\rho_n}\frac{dF_n(z_{n+1})}{dz} = \frac{1}{\rho_{n+1}}\frac{dF_{n+1}(z_{n+1})}{dz},\ n = 0,1..(N-2) \tag{8.9}$$

$$\frac{dF_{N-1}(z_N)}{dz} = 0 \tag{8.10}$$

The above boundary conditions specify the upper surface to be a pressure release surface, and the bottom surface a rigid surface. The remaining boundary conditions are continuity of pressure and the normal velocity between adjacent layers.

Equation 8.7 implies the following condition

$$A_0^- = -A_0^+ \tag{8.11}$$

on the depth function in the top layer. Equation 8.10 implies the following condition

$$A_{N-1}^- = +A_{N-1}^+ e^{+2ih_{N-1}d_{N-1}} \tag{8.12}$$

on the depth function in the bottom layer. The remaining boundary conditions imply the relationship:

$$A_{n+1} = M_n A_n \tag{8.13}$$

where A_n is the column vector

$$A_n = \begin{pmatrix} A_n^+ \\ A_n^- \end{pmatrix} \tag{8.14}$$

and M_n is the matrix

$$M_n = \begin{pmatrix} M_{11}^n & M_{12}^n \\ M_{21}^n & M_{22}^n \end{pmatrix} \tag{8.15}$$

$$M^{-1}{}_n = \frac{1}{\det(M_n)}\begin{pmatrix} M_{22}^n & -M_{21}^n \\ -M_{12}^n & M_{11}^n \end{pmatrix} \tag{8.16}$$

$$M_{11}^n = (1+\frac{\rho_{n+1}}{\rho_n}\frac{h_n}{h_{n+1}})e^{+ih_n d_n}/2 \tag{8.17}$$

$$M_{12}^n = (1-\frac{\rho_{n+1}}{\rho_n}\frac{h_n}{h_{n+1}})e^{-h_n d_n}/2 \tag{8.18}$$

$$M_{21}^n = (1-\frac{\rho_{n+1}}{\rho_n}\frac{h_n}{h_{n+1}}e^{+h_n d_n})/2 \tag{8.19}$$

$$M_{22}^n = (1+\frac{\rho_{n+1}}{\rho_n}\frac{h_n}{h_{n+1}})e^{-ih_n d_n}/2 \tag{8.20}$$

Here,

$$d_n = (z_{n+1} - z_n),\ n = 0,1...N-1 \tag{8.21}$$

is the thickness of the n'th layer.

The characteristic equation for the normal modes is the transcendental equation:

$$\frac{dF_n(z_N)}{dz} = ih_{N-1}(A_{N-1}^+ e^{+ih_{N-1}d_{N-1}} - A_{N-1}^- e^{-ih_{N-1}d_{N-1}}) = 0 \tag{8.22}$$

where the depth function in the bottom layer is determined by the following relationship between the depth function in the top layer with that in the bottom layer:

$$A_{N-1} = M_{N-2}...M_0 A_0 \tag{8.23}$$

$$A_0 = A_0^+\begin{pmatrix}+1\\-1\end{pmatrix} \tag{8.24}$$

The above relationship determines the depth functions throughout the waveguide up to the multiplicative factor A_0^+. This factor is determined by normalizing the depth functions according to the following condition:

$$\int_0^D \frac{F(z)F(z)}{\rho(z)}dz = \sum_{n=0} \int \frac{F_n(z)F_n(z)}{\rho_n}dz = 1 \tag{8.25}$$

Let $\{q_m\}$ be a complete basis for the normal modes in the complex q-plane, and $F(z:q_m)$ the corresponding depth function. Then the Green's Function for this waveguide is equal to the following summation:

$$G(r,z,z_s) = \frac{i}{4\rho(z_s)} \sum_m H_0^{(1)}(q_m r) F(z:q_m) F(z_s:q_m) \tag{8.26}$$

where the depth functions satisfy the following orthonormality condition:

$$\int_0^D \frac{F(z:q_m)F(z:q_n)}{\rho(z)} dz = \delta_m^n \tag{8.27}$$

An alternative method of computing the Green's Function is to directly evaluate the contour integral given in Equation 8.1. This is the approach used in fast field programs such as "SAFARI" and "OASES". This approach requires the construction of the depth dependent Green's Function off mass shell. It has the advantage that one doesn't have to solve the transcendental equation given by Equation 8.22 for the normal modes.

In order to construct the off mass shell solution for the depth dependent Green's Function one constructs the functions $p_1(z)$ and $p_2(z)$ according to the following prescription. First, define the quantities:

$$z_< = \min(z, z_s) \tag{8.28}$$

$$z_> = \max(z, z_s) \tag{8.29}$$

Suppose the coordinates $z_<$ and $z_>$ are located in the n and n' layers respectively, that is, These coordinates satisfy the inequalities:

$$z_n < z_< < z_{n+1} \tag{8.30}$$

$$z_{n'} < z_> < z_{n'+1} \tag{8.31}$$

Define the coefficients:

$$\begin{pmatrix} \vec{A}_n^+ \\ \vec{A}_n^- \end{pmatrix} = M_{n-1} \ldots M_0 \begin{pmatrix} +1 \\ -1 \end{pmatrix} \tag{8.32}$$

$$\begin{pmatrix} \vec{A}_{n'}^+ \\ \vec{A}_{n'}^- \end{pmatrix} = M_{n'}^{-1} \ldots M_{N-2}^{-1} \begin{pmatrix} +e^{-ih_{N-1}d_{N-1}} \\ +e^{+ih_{N-1}d_{N-1}} \end{pmatrix} \tag{8.33}$$

and define the functions:

$$p_1(z) = \bar{A}_n^+ F_n^+(z) + \bar{A}_n^- F_n^-(z) \tag{8.34}$$

$$p_2(z) = \bar{A}_{n'}^+ F_{n'}^+(z) + \bar{A}_{n'}^- F_{n'}^-(z) \tag{8.35}$$

These functions satisfy the homogeneous equation for the depth functions, and the boundary condition at the upper and lower surface, respectively. The depth dependent Green's Function is given by the following expression in terms of these two functions:

$$G_z(z, z_s : q) = -\frac{p_1(z_<)p_2(z_>)}{W(p_1, p_2)(z_s)} \tag{8.36}$$

where

$$W(p_1, p_2) = p_1(z)\frac{dp_2(z)}{dz} - \frac{dp_1(z)}{dz}p_2(z) \tag{8.37}$$

is the Wronskian of this pair of functions.

As a means of comparing the above contour integral representation with the normal mode representation of the Green's Function, the normal mode representation is given by the following expression in terms of the depth functions of the normal modes.

$$G_z(z, z_s : q) = \frac{1}{\rho(z_s)}\sum \frac{F(z : q_m)F(z_s : q_m)}{(q^2 - q_m^2)} \tag{8.38}$$

Substituting the above expression for the depth dependent Green's Function into Equation 8.1, one arrives at the normal mode expression for the Green's Function given in Equation 8.26.

9. N HOMOGENEOUS FLUID LAYERS OVER A HOMOGENEOUS HALF-SPACE

This section describes the modifications of the previous section, when the rigid bottom of the previous section is replaced by a homogeneous half-space.

One of the major differences between the case of a homogeneous half-space and a rigid bottom is that in the case of a half-space there are generally a finite number of normal modes, where as, in the case of a rigid bottom there are generally an infinite number of normal modes. In addition, the integrand of the equation

$$G(r,z,z_s)=\frac{1}{4\pi}\int_{-\infty+i0}^{+\infty-i0}H_0^{(1)}(qr)G_z(z,z_s:q)qdq \tag{9.1}$$

is generally not an even function of the vertical wavenumber of the homogeneous half-space. This results in a branch cut contribution of the spectral integral from the integration of the spectral integral around the branch cut of the vertical wavenumber of the homogeneous half-space in the complex q-plane. The half-space also affects the normalization of the normal modes.

Let $\{\rho_n,c_n : n=0,1..N-1\}$ denote the density and sound speed in the N layers. Let $\{z_n : n=0,1...N\}$ denote the z-coordinate of the N+1 interfaces. Without loss of generality, one assumes $z_0=0$, and $z_N=D$. Let $\{\rho_N,c_N\}$ denote the density and speed of sound in the homogeneous half-space.

The depth function in the n'th layer is required to satisfy the differential equation:

$$\frac{dF(z)}{dz}+(k_n^2-q^2)=0 \tag{9.2}$$

where $k_n=\omega/c_n$ is the wavenumber in the n'th layer. The functions:

$$F_n^+(z)=e^{+ih_n(z-z_n)} \tag{9.3}$$

$$F_n^-(z)=e^{-ih_n(z-z_n)} \tag{9.4}$$

denote a basis for the downward and upward going waved in the n'th layer, where

$$h_n(q)=i\sqrt{q^2-k_n^2} \tag{9.5}$$

is the vertical wavenumber in the n'th layer, and q is the horizontal wavenumber.

One may again represent the depth function in the n'th layer as the following sum

$$F_n(z) = A_n^+ F_n^+(z) + A_n^- F_n^-(z) \tag{9.6}$$

of upward and downward going waves. The depth function is required to satisfy the following boundary conditions:

$$F_0(z_0) = 0 \tag{9.7}$$

$$F_n(z_{n+1}) = F_{n+1}(z_{n+1}),\ n = 0,1..(N-1) \tag{9.8}$$

$$\frac{1}{\rho_n}\frac{dF_n(z_{n+1})}{dz} = \frac{1}{\rho_{n+1}}\frac{dF_{n+1}(z_{n+1})}{dz},\ n = 0,1..(N-1) \tag{9.9}$$

Assuming the source is above the homogeneous half-space, one requires the condition that the coefficient:

$$A_N^- = 0 \tag{9.10}$$

of the upward moving waves in the homogeneous half-space vanish.

Equation 9.7 implies the following condition

$$A_0^- = -A_0^+ \tag{9.11}$$

on the depth function in the top layer. Boundary conditions shown in Equations 9.8 and 9.9 imply the relationship,

$$A_{n+1} = M_n A_n \tag{9.12}$$

where A_n is the column vector

$$A_n = \begin{pmatrix} A_n^+ \\ A_n^- \end{pmatrix} \tag{9.13}$$

and M_n is the matrix given by Equations 8.15 through 8.20 in the previous section. Equation 9.10 in conjunction with Equations 9.8 and 9.9 imply one can replace the boundary condition for the rigid bottom by the following boundary condition:

$$f^B(q)F_{N-1}(z_N : q) + \frac{g^B(q)}{\rho_{N-1}}\frac{d}{dz}F_{N-1}(z_N : q) = 0 \tag{9.14}$$

where the coefficients $f^B(q)$ and $g^B(q)$ are given by the following expressions:

$$f^B(q) = \frac{-ih_N(q)}{\rho_N} \tag{9.15}$$

$$g^B(q) = 1 \tag{9.16}$$

The depth functions $\{F(z:q_m)\}$ are normalized by the following condition:

$$\int_0^D \frac{F(z:q_m)^2}{\rho(z)} dz + \frac{1}{2q_m}\frac{\partial}{\partial q}\left(\frac{f^B(q_m)}{g^B(q_m)}\right)F(D:q_m)^2 = 1 \tag{9.17}$$

The Green's Function is given by the following residue term and cut contribution:

$$G(r,z,z_s) = \frac{i}{4\rho(z_s)}\sum_m H_0^{(1)}(q_m r)F(z:q_m)F(z_s:q_m) + G_{cut}(r,z,z_s) \tag{9.18}$$

Here, the cut contribution is given by the following expression:

$$G_{cut}(r,z,z_s) = \frac{1}{8\pi}\int_{-\infty}^{+\infty} H_0^{(1)}(qr)\{G_z(z,z_s:+h_N) - G_z(z,z_s:-h_N)\}h_N dh_N \tag{9.19}$$

The above integral is an integral over the branch cut for the homogeneous half-space, where the integrand is proportional to the difference in the depth function across the cut. Evaluation of the above integral requires the construction of the off mass shell representation of the depth dependent Green's Function.

In order to construct the off mass shell solution for the depth dependent Green's Function one must construct the functions $p_1(z)$ and $p_2(z)$ according to the following prescription. First, define the quantities:

$$z_< = \min(z,z_s) \tag{9.20}$$

$$z_> = \max(z,z_s) \tag{9.21}$$

Suppose the coordinates $z_<$ and $z_>$ are located in the n and n' layers respectively, that is, these coordinates satisfy the inequalities:

$$z_n < z_< < z_{n+1} \tag{9.22}$$

$$z_{n'} < z_> < z_{n'+1} \tag{9.23}$$

Define the coefficients:

$$\begin{pmatrix} \bar{A}_n^+ \\ \bar{A}_n^- \end{pmatrix} = M_{n-1} \ldots M_0 \begin{pmatrix} +1 \\ -1 \end{pmatrix} \tag{9.24}$$

$$\begin{pmatrix} \bar{A}_{n'}^+ \\ \bar{A}_{n'}^- \end{pmatrix} = M_{n'}^{-1} \ldots M_{N-1}^{-1} \begin{pmatrix} 1 \\ 0 \end{pmatrix} \tag{9.25}$$

and define the functions:

$$p_1(z) = \bar{A}_n^+ F_n^+(z) + \bar{A}_n^- F_n^-(z) \tag{9.26}$$

$$p_2(z) = \bar{A}_{n'}^+ F_{n'}^+(z) + \bar{A}_{n'}^- F_{n'}^-(z) \tag{9.27}$$

These functions satisfy the homogeneous equation for the depth functions, and the boundary condition at the upper and lower surface, respectively. The depth dependent Green's Function is given by the following expression in terms of these two functions:

$$G_z(z, z_s : q) = -\frac{p_1(z_<) p_2(z_>)}{W(p_1, p_2)(z_s)} \tag{9.28}$$

where

$$W(p_1, p_2) = p_1(z)\frac{dp_2(z)}{dz} - \frac{dp_1(z)}{dz} p_2(z) \tag{9.29}$$

is the Wronskian of this pair of functions.

10. N INHOMOGENEOUS FLUID LAYERS

This section describes the modifications of the previous two sections in the case the index of refraction squared is a piece-wise linear function of depth. In this case, the complex exponential functions of the previous sections are replaced by complex Airy functions. In the case of an inhomogeneous layer, there is no longer a natural split of the depth functions into upward moving and downward moving components. This work is based on the description by Stickler[7] on the computation of normal modes in a waveguide with a piece-wise linear index of refraction squared.

Let $\{\rho_n, c_n^T, c_n^B : n = 0,1...(N-1)\}$ denote the density, sound speed at the top of the layer, and sound speed at the bottom of the layer for the N inhomogeneous layers, where the index of refraction squared is the following linear function of depth:

$$c_n^{-2}(z) = \alpha_n - \beta_n(z - z_n) \tag{10.1}$$

$$\alpha_n = (c_n^T)^{-2} \tag{10.2}$$

$$\beta_n = ((c_n^T)^{-2} - (c_n^T)^{-2}) / d_n \tag{10.3}$$

Define the change of coordinates:

$$z \to Z_n(z) \tag{10.4}$$

in the n'th layer, where the coordinate function $Z_n(z)$ is required to satisfy the following constraint:

$$Z_n(z)(\frac{dZ_n(z)}{dz})^{-2} = -(\omega^2 / c(z)^2 - q^2) \tag{10.5}$$

The coordinate function $Z_n(z)$ is given by the following linear function of z:

$$Z_n(z) = -L_n^2(\omega^2(\alpha_n - \beta_n(z - z_n)) - q^2) \tag{10.6}$$

where the coefficient L_n is given by the expression

$$L_n = (\omega^4 \beta_n^2)^{-1/6} \tag{10.7}$$

The derivative of this coordinate function is given by the expression:

$$\frac{dZ_n(z)}{dz} = \omega^2 \beta_n L_n^2 \tag{10.8}$$

The differential equation for the depth function

$$\frac{dF_n(z)}{dz} + (k_n^2(z) - q^2)F_n(z) = 0 \tag{10.9}$$

in the n'th layer is transformed into the following differential equation:

$$\frac{dF_n}{dZ_n} - Z_n F_n = 0 \tag{10.10}$$

under the coordinate transformation $z \to Z_n$. Equation 10.10 is the Airy Differential Equation, which implies the depth function is given by the following linear combination of Airy Functions:

$$F_n(z) = A_n^+ Ai(Z_n(z)) + A_n^- Bi(Z_n(z)) \tag{10.11}$$

Note the above expression for the depth function becomes ill conditioned in the limit the sound speed gradient in the layer vanishes, that is, the coefficient $\beta_n \to 0$ approaches zero. In this case the coordinate transformation $z \to Z_n$ becomes ill defined.

The ill-conditioned nature of this coordinate transformation for infinitesimally small sound speed gradients can lead to numerical overflow errors in the computation of the depth functions. Therefore it is necessary to place a minimum limit on the sound speed gradient for which the above linear representation of the index of refraction squared will be used. If the sound speed gradient is below this minimum, the sound speed profile in the layer is considered to be a constant, and the depth functions of the previous two sections are used.

Thus one introduces the following basis for the depth functions if the sound speed gradient is sufficiently small:

$$F_n^+(z) = e^{+ih_n(z-z_n)} \tag{10.12}$$

$$F_n^-(z) = e^{-ih_n(z-z_n)} \tag{10.13}$$

where the sound speed

$$c_n = (c_n^T + c_n^B)/2 \tag{10.14}$$

is equal to the average of the speed of sound at the top and bottom surface, and

$$h_n(q) = i\sqrt{q^2 - k_n^2} \tag{10.15}$$

is the vertical wavenumber. Otherwise, we use the following basis:

$$F_n^+(z) = Ai(Z_n(z)) \tag{10.16}$$

$$F_n^-(z) = Bi(Z_n(z)) \tag{10.17}$$

for the depth functions, provided the gradient β_n is sufficiently large.

Let

$$W_n(z:q) = F_n^+(z)\frac{dF_n^-(z)}{dz} - \frac{dF_n^+(z)}{dz}F_n^-(z) \tag{10.18}$$

denote the Wronskian of the above basis for the depth functions. In the case of vanishing gradient (iso-velocity layer), this Wronskian is given by the expression:

$$W_n(z:q) = -2ih_n(q) \tag{10.19}$$

where $h_n(q)$ is the vertical wavenumber of this iso-velocity layer. Otherwise, the Wronskian is equal to the expression:

$$W_n(z:q) = \omega^2 \beta_n L_n^{-2} / \pi \tag{10.20}$$

where the expression

$$Ai(x)\frac{dBi(x)}{dx} - \frac{dAi(x)}{dx}Bi(x) = \frac{1}{\pi} \tag{10.21}$$

is used for the Wronskian of the Airy Functions.

The depth function $F_n(z:q)$ in the n'th layer may be represented by the following linear combination of the functions $F_n^+(z:q)$ and $F_n^-(z:q)$:

$$F_n(z:q) = A_n^+ F_n^+(z:q) + A_n^- F_n^-(z:q) \tag{10.22}$$

The depth function in the top layer satisfies the following constraint for a pressure release surface:

$$F_0(z_0) = 0 \tag{10.23}$$

In addition, the depth function must satisfy the constraint of continuity of pressure and normal velocity at the interface between adjacent layers:

$$F_n(z_{n+1}) = F_{n+1}(z_{n+1}),\ n = 0,1..(N-1) \tag{10.24}$$

$$\frac{1}{\rho_n}\frac{dF_n(z_{n+1})}{dz} = \frac{1}{\rho_{n+1}}\frac{dF_{n+1}(z_{n+1})}{dz},\ n = 0,1..(N-1) \tag{10.25}$$

In the case of a rigid basement, the depth function satisfies the boundary condition:

$$\frac{dF_{N-1}(z_N)}{dz} = 0 \tag{10.26}$$

at the bottom surface. In the case of a homogeneous half-space, the coefficient for an upward moving wave in the basement is required to vanish and the depth function is required to satisfy the constraints of continuity of pressure and normal velocity at the interface with the homogeneous half-space:

$$A_N^- = 0 \tag{10.27}$$

$$F_{N-1}(z_N) = F_N(z_N) \tag{10.28}$$

$$\frac{1}{\rho_{N-1n}}\frac{dF_{N-1}(z_N)}{dz} = \frac{1}{\rho_N}\frac{dF_N(z_N)}{dz} \tag{10.29}$$

The constraint on the depth function by the presence of the bottom can be reformulated by a boundary condition of the following form:

$$f^B(q)F_{N-1}(z_N:q) + \frac{g^B(q)}{\rho_{N-1}}\frac{d}{dz}F_{N-1}(z_N:q) = 0 \tag{10.30}$$

where the coefficients $f^B(q)$ and $g^B(q)$ are given by the following expressions in the case of a homogeneous half-space:

$$f^B(q) = \frac{-ih_N(q)}{\rho_N} \tag{10.31}$$

$$g^B(q) = 1 \tag{10.32}$$

and the following expressions in the case of a rigid bottom:

$$f^B(q) = 0 \tag{10.33}$$

$$g^B(q) = 1 \tag{10.34}$$

The depth functions $\{F(z:q_m)\}$ are normalized by the following condition:

$$\int_0^D \frac{F(z:q_m)^2}{\rho(z)} dz + \frac{1}{2q_m}\frac{\partial}{\partial q}\left(\frac{f^B(q_m)}{g^B(q_m)}\right)F(D:q_m)^2 = 1 \tag{10.35}$$

Equation 10.23 implies the following relationship for the coefficients of the depth function in the top layer:

$$A_0^- = -A_0^+ \tag{10.36}$$

Equations 10.24 and 10.25 imply the following relationship between the coefficients of adjacent layers:

$$A_{n+1} = M_n A_n \tag{10.37}$$

Here A_n is the coefficient vector

$$A_n = \begin{pmatrix} A_n^+ \\ A_n^- \end{pmatrix} \tag{10.38}$$

and M_n is the propagator matrix

$$M_n = \begin{pmatrix} M_{11}^n & M_{12}^n \\ M_{21}^n & M_{22}^n \end{pmatrix} \tag{10.39}$$

whose matrix elements are given by the following relations:

$$M_{11}^m = +\frac{1}{W_{n+1}(z_{n+1})}\{F_n^+(z_{n+1})\frac{dF_{n+1}^-(z_{n+1})}{dz} - \frac{\rho_{n+1}}{\rho_n}\frac{dF_n^+(z_{n+1})}{dz}F_{n+1}^-(z_{n+1})\} \tag{10.40}$$

$$M_{12}^m = +\frac{1}{W_{n+1}(z_{n+1})}\{F_n^-(z_{n+1})\frac{dF_{n+1}^-(z_{n+1})}{dz} - \frac{\rho_{n+1}}{\rho_n}\frac{dF_n^-(z_{n+1})}{dz}F_{n+1}^-(z_{n+1})\} \tag{10.41}$$

$$M_{21}^m = -\frac{1}{W_{n+1}(z_{n+1})}\{F_n^+(z_{n+1})\frac{dF_{n+1}^+(z_{n+1})}{dz} - \frac{\rho_{n+1}}{\rho_n}\frac{dF_n^+(z_{n+1})}{dz}F_{n+1}^+(z_{n+1})\} \tag{10.42}$$

$$M_{22}^{m} = -\frac{1}{W_{n+1}(z_{n+1})}\{F_n^-(z_{n+1})\frac{dF_{n+1}^+(z_{n+1})}{dz} - \frac{\rho_{n+1}}{\rho_n}\frac{dF_n^-(z_{n+1})}{dz}F_{n+1}^+(z_{n+1})\} \tag{10.43}$$

Recall that the expression:

$$W_n(z:q) = F_n^+(z)\frac{dF_n^-(z)}{dz} - \frac{dF_n^+(z)}{dz}F_n^-(z) \tag{10.44}$$

is the Wronskian of the basis for the depth functions in the n'th layer.

Using Equations 10.36 and 10.37, the depth functions in all the layers are determined up to a common multiplicative factor. However, the depth functions constructed from Equations 10.36 and 10.37 do not necessarily satisfy the boundary condition in Equation 10.30 at the bottom layer. This equation acts as the characteristic equation, whose solutions determine the location of the normal modes in the complex q-plane.

Given the above construction of the coefficients of the depth function, the construction of the depth dependent Green's Function and the spectral integral proceeds in analogy to the previous two sections.

11. EXAMPLE: ISO-VELOCITY LAYER OVER A RIGID BOTTOM

This section describes an iso-velocity layer over a rigid bottom.

The characteristic equation for the normal modes in this case is given by the equation:

$$\cos(h_0 d) = 0 \tag{11.1}$$

where

$$h_0 = i\sqrt{q^2 - k_0^2} \tag{11.2}$$

is the vertical wavenumber. The solutions of this equation are given by the modes vertical wavenumber:

$$h_0(q_m) = (m + 1/2)\frac{\pi}{d} \tag{11.3}$$

The normalized depth function is given by the following relationship:

$$F(z : q_m) = \sqrt{\frac{2\rho_0}{d}} \sin(h_0(q_m)d) \tag{11.4}$$

The Green's Function is given by the following modal summation:

$$G(r, z, z_s) = \frac{i}{2d}\sum_m H_0^{(1)}(q_m r)\sin(h_0(q_m)z)\sin(h_0(q_m)z_s) \tag{11.5}$$

12. EXAMPLE: PEKERIS MODEL

This section describes the Pekeris Model, that is, an iso-velocity layer over a homogeneous half-space.

The normal modes for the Pekeris Model are given by the solutions of the characteristic equation:

$$mh_0 \cos(h_0 d) - ih_1 \sin(h_0 d) = 0 \tag{12.1}$$

where m is the density ratio.

$$m = \frac{\rho_1}{\rho_0} \tag{12.2}$$

The depth dependent Green's Function is given by the following expression:

$$G_z(z,z_s) = \frac{\sin(h_0 z_<)}{h_0} \frac{(mh_0 \cos(h_0(d-z_>)) - ih_1 \sin(h_0(d-z_>)))}{(mh_0 \cos(h_0(d)) - ih_1 \sin(h_0(d)))} \tag{12.3}$$

The depth function in the top layer is given by the following expression:

$$F(z:q) = A_0^+ \sin(h_0 z) \tag{12.4}$$

where the normalization factor is given by the expression.

$$A_0^+ = \frac{mh_0}{\sin(h_0 d)} \sqrt{\frac{2\rho(z_s)h_1}{h_1 d(m^2 h_0^2 - h_1^2) + im(h_0^2 - h_1^2)}} \tag{12.5}$$

The normal mode portion of the Green's Function is given by the following sum of residues:

$$G_{Mode}(r,z,z_s) =$$

$$\frac{i}{2}\sum_j H_0^{(1)}(q_j r)\sin(h_0 z)\sin(h_0 z_s)\left(\frac{mh_0}{\sin(h_0 d)}\right)^2 \frac{h_1}{h_1 d(m^2 h_0^2 - h_1^2) + im(h_0^2 - h_1^2)} \tag{12.6}$$

The branch cut contribution to the Green's Function is given by the following spectral integral:

$$G_{Cut}(r,z,z_s) = +\frac{im}{4\pi}\int_{-\infty}^{+\infty} h_1^2\, dh_1 H_0^{(1)}(qr)\frac{\sin(h_0 z)\sin(h_0 z_s)}{m^2 h_0^2 \cos(h_{0d})^2 + h_1^2 \sin(h_0 d)^2} \tag{12.7}$$

13. EFFECT OF A SMALL WAVE-HEIGHT, RANDOMLY ROUGH, TWO FLUID INTERFACE ON THE COHERENT COMPONENT OF THE REFLECTED AND TRANSMITTED FIELD

This section describes the effects of a randomly rough two-fluid interface with small wave-height on the coherent component of the reflected and transmitted field[8].

Let $\{\rho_1, c_1\}$ and $\{\rho_2, c_2\}$ denote the density and sound speed in Regions One and Two, where the plane z = 0 is the interface between Region One (z > 0) and Region Two (z < 0) in the unperturbed case. The pressure field in Regions One and Two are solutions of the differential equations:

$$\nabla^2 p_1 + k_1^2 p_1 = 0 \tag{13.1}$$

$$\nabla^2 p_2 + k_2^2 p_2 = 0 \tag{13.2}$$

where p_1 and p_2 are the pressure fields in Regions One and Two respectively. Here, k_1 and k_2 are the wavenumber in Regions One and Two respectively. The pressure field is subject to the following boundary conditions at the interface z =0.

$$p_1(z=0) = p_2(z=0) \tag{13.3}$$

$$\frac{1}{\rho_1}\frac{dp_1(z=0)}{dz} = \frac{1}{\rho_2}\frac{dp_2(z=0)}{dz} \tag{13.4}$$

Let

$$p_{inc} = \exp[+i\vec{k}_{0,\perp} \bullet \vec{r} - ik_{10,z} z] \tag{13.5}$$

be the incident field, where $\vec{k}_{0,\perp}$ is the horizontal wave-vector of the incident field in the xy-plane, and

$$k_{10,z} = i\sqrt{k_{0,\perp}^2 - k_1^{\,2}} \tag{13.6}$$

is the vertical wavenumber of the incident field. The unperturbed solution for the pressure field in the two regions is given by the following expressions:

$$p_1 = \exp[+i\vec{k}_{0,\perp} \bullet \vec{r} - ik_{10,z} z] + R_0 \exp[+i\vec{k}_{0,\perp} \bullet \vec{r} + ik_{10,z} z] \tag{13.7}$$

$$p_1 = T_0 \exp[+i\vec{k}_{0,\perp} \bullet \vec{r} - ik_{20,z} z] \tag{13.8}$$

where

$$R_0 = \frac{k_{10,z} / \rho_1 - k_{20,z} / \rho_2}{k_{10,z} / \rho_1 + k_{20,z} / \rho_2} \tag{13.9}$$

$$T_0 = 1 + R_0 = \frac{2k_{10,z} / \rho_1}{k_{20,z} / \rho_2 + k_{20,z} / \rho_2} \tag{13.10}$$

are the reflection and transmission coefficients of the interface.

Consider the case that the interface is a random rough surface given by the equation:

$$z = \alpha(r) \tag{13.11}$$

where $\alpha(r)$ is a random variable with zero mean and slope.

$$< \alpha >= 0 \tag{13.12}$$

$$< \nabla_\perp \alpha >= 0 \tag{13.13}$$

The pressure field is required to satisfy the boundary conditions:

$$p_1(z = \alpha) = p_2(z = \alpha) \tag{13.14}$$

$$\frac{1}{\rho_1} \frac{\partial p_1(z = \alpha)}{\partial n} = \frac{1}{\rho_2} \frac{\partial p_2(z = \alpha)}{\partial n} \tag{13.15}$$

where

$$\frac{\partial}{\partial n} = \nabla_n = \frac{1}{\sqrt{1 + \nabla_\perp \alpha \bullet \nabla_\perp \alpha}} (-\vec{\nabla}_\perp \alpha \bullet \vec{\nabla} + \frac{\partial}{\partial z}) \tag{13.16}$$

is the derivative in the direction of the normal to the rough surface:

$$\vec{n} = \frac{1}{\sqrt{1 + \nabla_\perp \alpha \bullet \nabla_\perp \alpha}} (-\frac{\partial \alpha}{\partial x}, -\frac{\partial \alpha}{\partial y}, 1) \tag{13.17}$$

Let us express the pressure field in Regions One and Two as the sum of the mean field and a zero mean stochastic field.

$$p_1 =< p_1 > +w_1 \tag{13.18}$$

$$p_2 =< p_2 > +w_2 \tag{13.19}$$

Expanding the fields about the mean surface, one arrives at the following boundary conditions on the mean surface (z = 0):

$$p_1 + \alpha \frac{\partial p_1}{\partial z} + \frac{\alpha^2}{2}\frac{\partial p_1}{\partial z} = p_2 + \alpha \frac{\partial p_2}{\partial z} + \frac{\alpha^2}{2}\frac{\partial p_2}{\partial z} \tag{13.20}$$

$$\frac{1}{\rho_1}\left[\frac{\partial p_1}{\partial z} + \alpha \frac{\partial^2 p_1}{\partial z^2} + \frac{\alpha^2}{2}\frac{\partial^3 p_1}{\partial z^3} - \vec{\nabla}_\perp \alpha \bullet \vec{\nabla} p_1\right] =$$

$$\frac{1}{\rho_2}\left[\frac{\partial p_2}{\partial z} + \alpha \frac{\partial^2 p_2}{\partial z^2} + \frac{\alpha^2}{2}\frac{\partial^3 p_2}{\partial z^3} - \vec{\nabla}_\perp \alpha \bullet \vec{\nabla} p_2\right] \tag{13.21}$$

Here, one has only kept terms up to quadratic terms in the random variable. Taking the ensemble average of the above equations, one obtains the following equations for the mean field on the mean surface (z = 0):

$$< p_1 > - < p_2 >= F_1 \tag{13.22}$$

$$\frac{1}{\rho_1}\frac{\partial < p_1 >}{\partial z} - \frac{1}{\rho_2}\frac{\partial < p_2 >}{\partial z} = F_2 \tag{13.23}$$

The source terms on the right hand side are given by the following expressions quadratic in the random variable:

$$F_1 = - < \alpha\left[\frac{\partial w_1}{\partial z} - \frac{\partial w_2}{\partial z}\right] > - \frac{< \alpha^2 >}{2}\left[\frac{\partial^2 < p_1 >}{\partial z^2} - \frac{\partial^2 < p_2 >}{\partial z^2}\right] \tag{13.24}$$

$$F_2 = - < \alpha\left[\frac{1}{\rho_1}\frac{\partial^2 w_1}{\partial z^2} - \frac{1}{\rho_2}\frac{\partial^2 w_2}{\partial z^2}\right] > - \frac{< \alpha^2 >}{2}\left[\frac{1}{\rho_1}\frac{\partial^3 < p_1 >}{\partial z^3} - \frac{1}{\rho_2}\frac{\partial^3 < p_2 >}{\partial z^3}\right] +$$

$$< \vec{\nabla}_\perp \alpha \bullet \vec{\nabla}\left[\frac{w_1}{\rho_1} - \frac{w_2}{\rho_2}\right] > \tag{13.25}$$

Note, the source terms are quadratic in the random variable. Subtracting Equations 13.24 and 13.25 from Equations 13.20 and 13.21, one arrives at the following boundary conditions for the stochastic field on the mean surface (z = 0).

$$w_1 - w_2 = G_1 \tag{13.26}$$

$$\frac{1}{\rho_1}\frac{\partial w_1}{\partial z} - \frac{1}{\rho_2}\frac{\partial w_2}{\partial z} = G_2 \tag{13.27}$$

The source terms on the right hand side of the boundary conditions are given by the following expressions linear in the random variable:

$$G_1 = -\alpha\left[\frac{\partial < p_1 >}{\partial z} - \frac{\partial < p_2 >}{\partial z}\right] \tag{13.28}$$

$$G_2 = -\alpha\left[\frac{1}{\rho_1}\frac{\partial^2 < p_1 >}{\partial z^2} - \frac{1}{\rho_2}\frac{\partial^2 < p_2 >}{\partial z^2}\right] + \vec{\nabla}_\perp \alpha \bullet \vec{\nabla}\left[\frac{< p_1 >}{\rho_1} - \frac{< p_2 >}{\rho_2}\right] \tag{13.29}$$

In order to solve Equations 13.22 and 13.23 for the mean field, one must eliminate the stochastic field from the right-hand side of these equations. In order to do this, one must first solve Equations 13.26 and 13.27 for the stochastic fields in terms of the mean field and the random wave height. One may solve Equations 13.26 and 13.27 by taking the Fourier Transform of these equations. This enables us to reduce the solution of these equations into a set of algebraic equations for the spectral strength of these fields.

Let us introduce the following Fourier decomposition of the random wave height:

$$\alpha(r) = \frac{1}{2\pi}\int d^2\xi_\perp \tilde{\alpha}(\xi) e^{+i\vec{\xi}_\perp \bullet \vec{r}} \tag{13.30}$$

Assume the cross correlation function of this random variable is homogeneous. Then the cross correlation function depends only upon the difference of the two points, and is of the following form:

$$< \alpha(r)\alpha(r') >=< \alpha^2 > \sigma(r - r') =< \alpha^2 > \frac{1}{2\pi}\int d^2\xi_\perp S(\xi) e^{+i\vec{\xi}_\perp \bullet \vec{r}} \tag{13.31}$$

where $< \alpha^2 >$ is the root mean square wave height squared, and $S(\xi)$ is the power spectrum of the surface. The following ensemble averages follow from the above definition of the power spectrum:

$$< \alpha(r)\alpha(\eta) >=< \alpha^2 > S(\eta) e^{+i\vec{\eta}_\perp \bullet \vec{r}} \tag{13.32}$$

$$< \alpha(\xi)\alpha(\eta) >= 2\pi\delta(\xi_\perp - \eta_\perp) < \alpha^2 > S(\eta) \tag{13.33}$$

The Fourier decompositions of the stochastic fields in Regions One and Two have the following form:

$$w_1(r,z)=\frac{1}{2\pi}\int d^2\xi_\perp e^{+i\vec{\xi}_\perp \bullet \vec{r}+i\xi_{1,z}z}\tilde{w}_1(\xi) \tag{13.34}$$

$$w_2(r,z)=\frac{1}{2\pi}\int d^2\xi_\perp e^{+i\vec{\xi}_\perp \bullet \vec{r}-i\xi_{2,z}z}\tilde{w}_2(\xi) \tag{13.35}$$

The functions G_1 and G_2 are linear functionals of the random variable that have a Fourier decomposition of the following form:

$$G_1(r)=\frac{1}{(2\pi)^2}\int d^2\xi_\perp \int d^2\eta_\perp e^{+i\vec{\xi}_\perp \bullet \vec{r}}\tilde{\alpha}(\xi-\eta)\tilde{g}_1(\xi,\eta) \tag{13.36}$$

$$G_2(r)=\frac{1}{(2\pi)^2}\int d^2\xi_\perp \int d^2\eta_\perp e^{+i\vec{\xi}_\perp \bullet \vec{r}}\tilde{\alpha}(\xi-\eta)\tilde{g}_2(\xi,\eta) \tag{13.37}$$

The Fourier coefficients $g_1(\xi,\eta)$ and $g_2(\xi,\eta)$ have the following form in terms of the Fourier Transform of the mean field and its derivatives:

$$g_1(\xi,\eta)=-\left[\frac{\partial<\tilde{p}_1>}{\partial z}-\frac{\partial<\tilde{p}_2>}{\partial z}\right](\eta) \tag{13.38}$$

$$g_2(\xi,\eta)=$$

$$-\left[\frac{1}{\rho_1}\frac{\partial^2<\tilde{p}_1>}{\partial z^2}-\frac{1}{\rho_2}\frac{\partial^2<\tilde{p}_2>}{\partial z^2}\right](\eta)-\vec{\eta}_\perp \bullet(\vec{\xi}_\perp-\vec{\eta}_\perp)\left[\frac{<\tilde{p}_1>}{\rho_1}-\frac{<\tilde{p}_2>}{\rho_2}\right](\eta) \tag{13.39}$$

Substituting Equations 13.34 through 13.37 into Equations 13.26 and 13.27, one arrives at the following algebraic equations for the Fourier coefficients of the stochastic field:

$$\tilde{w}_1(\xi)-\tilde{w}_2(\xi)=\frac{1}{2\pi}\int d^2\eta_\perp \alpha(\xi-\eta)g_1(\xi,\eta) \tag{13.40}$$

$$\frac{i\xi_{1,z}}{\rho_1}\tilde{w}_1(\xi)-\frac{i\xi_{2,z}}{\rho_2}\tilde{w}_2(\xi)=\frac{1}{2\pi}\int d^2\eta_\perp \alpha(\xi-\eta)g_2(\xi,\eta) \tag{13.41}$$

Solution of the above equations leads to the following expression for the Fourier coefficients for the stochastic field:

$$\tilde{w}_1(\xi) = \frac{1}{2\pi}\int d^2\eta_\perp \frac{\alpha(\xi-\eta)}{i\xi_{1,z}/\rho_1 + i\xi_{2,z}/\rho_2}\left[\frac{i\xi_{2,z}}{\rho_2}g_1(\xi,\eta) + g_2(\xi,\eta)\right] \tag{13.42}$$

$$\tilde{w}_2(\xi) = \frac{1}{2\pi}\int d^2\eta_\perp \frac{\alpha(\xi-\eta)}{i\xi_{1,z}/\rho_1 + i\xi_{2,z}/\rho_2}\left[-\frac{i\xi_{1,z}}{\rho_1}g_1(\xi,\eta) + g_2(\xi,\eta)\right] \tag{13.43}$$

Substituting the above expressions for the stochastic field into Equations 13.24 and 13.25 for the source terms of the mean field, one obtains the following expressions:

$$\left[\xi_{1,z}\xi_{2,z}g_1(\xi,\eta)\left(\frac{1}{\rho_2}-\frac{1}{\rho_1}\right) - i(\xi_{1,z}+\xi_{2,z})g_2(\xi,\eta)\right]$$

$$F_1 = \frac{<\alpha^2>}{(2\pi)^2}\int d^2\xi_\perp \int d^2\eta_\perp e^{+i\vec{\eta}_\perp \bullet \vec{r}} \frac{S(\xi-\eta)}{i\xi_{1,z}/\rho_1 + i\xi_{2,z}/\rho_2}\times$$

$$-\frac{<\alpha^2>}{2}\frac{\partial^2}{\partial z^2}(<p_1> - <p_2>) \tag{13.44}$$

$$F_2 = \frac{<\alpha^2>}{(2\pi)^2}\int d^2\xi_\perp \int d^2\eta_\perp e^{+i\vec{\eta}_\perp \bullet \vec{r}} \frac{S(\xi-\eta)}{i\xi_{1,z}/\rho_1 + i\xi_{2,z}/\rho_2}\times$$

$$\left[\frac{i\xi_{1,z}\xi_{2,z}}{\rho_1\rho_2}(\xi_{1,z}+\xi_{2,z})g_1(\xi,\eta) + \left(\frac{\xi_{1,z}^2}{\rho_1}-\frac{\xi_{2,z}^2}{\rho_2}\right)g_2(\xi,\eta)\right.$$

$$\left. +\vec{\xi}_\perp \bullet (\vec{\xi}_\perp - \vec{\eta}_\perp)\left(\frac{i(\xi_{1,z}+\xi_{2,z})}{\rho_1\rho_2}g_1(\xi,\eta) + \left(\frac{1}{\rho_1}-\frac{1}{\rho_2}\right)g_2(\xi,\eta)\right)\right]$$

$$-\frac{<\alpha^2>}{2}\frac{\partial^3}{\partial z^3}\left(\frac{<p_1>}{\rho_1}-\frac{<p_2>}{\rho_2}\right) \tag{13.45}$$

To lowest order, the mean field is given by the unperturbed solution given below:

$$<p_{1,0}> = \exp[+i\vec{k}_{0,\perp} \bullet \vec{r}]\{\exp[-ik_{10,z}z] + R_0\exp[+ik_{10,z}z]\} \tag{13.46}$$

$$<p_{2,0}> = T_0\exp[+i\vec{k}_{0,\perp} \bullet \vec{r} - ik_{20,z}z] \tag{13.47}$$

$\vec{k}_{0,\perp}$ is the horizontal wave-vector of the incident field, and

$$k_{10,z} = i\sqrt{\vec{k}_{0,\perp} \bullet \vec{k}_{0,\perp} - k_1^2} \tag{13.48}$$

is the vertical wavenumber in Region One, and

$$k_{20,z} = i\sqrt{\vec{k}_{0,\perp} \bullet \vec{k}_{0,\perp} - k_2^2} \tag{13.49}$$

is the vertical wavenumber in Region Two. R_0 and T_0 are the reflection and transmission coefficients of the incident wave:

$$R_0 = \frac{k_{10,z}/\rho_1 - k_{20,z}/\rho_2}{k_{10,z}/\rho_1 + k_{20,z}/\rho_2} \tag{13.50}$$

$$T_0 = (1+R_0) = \frac{2k_{10,z}/\rho_1}{k_{10,z}/\rho_1 + k_{20,z}/\rho_2} \tag{13.51}$$

The Fourier coefficients of the unperturbed solutions are given by the following expressions:

$$<\tilde{p}_{1,0}>(\xi) = 2\pi\delta(\xi_\perp - k_{0,\perp})\{\exp[-ik_{10,z}z] + R_0 \exp[+ik_{10,z}z]\} \tag{13.52}$$

$$<\tilde{p}_{2,0}>(\xi) = 2\pi\delta(\xi_\perp - k_{0,\perp})T_0 \exp[-ik_{20,z}z] \tag{13.53}$$

Since the mean field and stochastic field are accurate up to terms quadratic order in the random wave height, one can substitute the unperturbed solution for the mean field in Equations 13.38 and 13.39 for the stochastic field source terms. This substitution leads to the following simplification:

$$g_1(\xi,\eta) = 2\pi\delta(\eta_\perp - k_{0,\perp})\tilde{g}_1(\xi) \tag{13.54}$$

$$g_2(\xi,\eta) = 2\pi\delta(\eta_\perp - k_{0,\perp})\tilde{g}_2(\xi) \tag{13.55}$$

$$\tilde{g}_1(\xi) = ik_{10,z}(1-R_0) - ik_{20,z}(1+R_0) \tag{13.56}$$

$$\tilde{g}_2(\xi) = (1+R_0)\left[\frac{k^2{}_{10,z}}{\rho_1} - \frac{k^2{}_{20,z}}{\rho_2} - \vec{k}_{0,\perp} \bullet (\vec{\xi}_\perp - \vec{k}_{0,\perp})(\frac{1}{\rho_1} - \frac{1}{\rho_2})\right] \tag{13.57}$$

Similarly, one may substitute the above results and the unperturbed solution into Equations 13.44 and 13.45 for the source terms of the mean field without loss of accuracy.

$$F_1 = \frac{<\alpha^2>}{2\pi} \exp[i\vec{k}_{0,\perp} \bullet \vec{r}] \int d^2\xi_\perp \frac{S(\xi_\perp - k_{0,\perp})}{i\xi_{1,z}/\rho_1 + i\xi_{2,z}/\rho_2} \left[\xi_{1,z}\xi_{2,z}\tilde{g}_1(\xi)(\frac{1}{\rho_2} - \frac{1}{\rho_1}) - i(\xi_{1,z} + \xi_{2,z})\tilde{g}_2(\xi) \right]$$

$$+ \frac{<\alpha^2>}{2} \exp[i\vec{k}_{0,\perp} \bullet \vec{r}](1 + R_0)(k_{1,z}^2 - k_{2,z}^2) \qquad (13.58)$$

$$F_2 = \frac{<\alpha^2>}{2\pi} \exp[i\vec{k}_{0,\perp} \bullet \vec{r}] \int d^2\xi_\perp \frac{S(\xi_\perp - k_{0,\perp})}{i\xi_{1,z}/\rho_1 + i\xi_{2,z}/\rho_2} \times$$

$$\left[\frac{i\xi_{1,z}\xi_{2,z}}{\rho_1\rho_2}(\xi_{1,z} + \xi_{2,z})\tilde{g}_1(\xi) + \left(\frac{\xi_{1,z}^2}{\rho_1} - \frac{\xi_{2,z}^2}{\rho_2} \right) \tilde{g}_2(\xi) \right.$$

$$\left. + \vec{\xi}_\perp \bullet (\vec{\xi}_\perp - \vec{k}_{0,\perp}) \left(\frac{i(\xi_{1,z} + \xi_{2,z})}{\rho_1\rho_2} \tilde{g}_1(\xi) + \left(\frac{1}{\rho_1} - \frac{1}{\rho_2} \right) \tilde{g}_2(\xi) \right) \right]$$

$$- i\frac{<\alpha^2>}{2} \exp[i\vec{k}_{0,\perp} \bullet \vec{r}] \left(\frac{k^3{}_{10,z}}{\rho_1}(1 - R_0) - \frac{k^3{}_{20,z}}{\rho_2}(1 + R_0) \right) \qquad (13.59)$$

In the above equations one notes that Equations 13.58 and 13.59 may be expressed in the following form proportional to the root mean square (rms) wave height and incident field evaluated at the unperturbed surface.

$$F_1 = <\alpha^2> p_{inc}(z=0)\hat{F}_1 \qquad (13.60)$$

$$F_2 = <\alpha^2> p_{inc}(z=0)\hat{F}_2 \qquad (13.61)$$

$$\hat{F}_1 = \frac{1}{2\pi} \int d^2\xi_\perp \frac{S(\xi_\perp - k_{0,\perp})}{i\xi_{1,z}/\rho_1 + i\xi_{2,z}/\rho_2} \left[\xi_{1,z}\xi_{2,z}\tilde{g}_1(\xi)(\frac{1}{\rho_2} - \frac{1}{\rho_1}) - i(\xi_{1,z} + \xi_{2,z})\tilde{g}_2(\xi) \right]$$

$$+ \frac{1}{2}(1 + R_0)(k_{1,z}^2 - k_{2,z}^2) \qquad (13.62)$$

$$\hat{F}_2 = \frac{1}{2\pi}\int d^2\xi_\perp \frac{S(\xi_\perp - k_{0,\perp})}{i\xi_{1,z}/\rho_1 + i\xi_{2,z}/\rho_2} \times \left[\frac{i\xi_{1,z}\xi_{2,z}}{\rho_1\rho_2}(\xi_{1,z}+\xi_{2,z})\tilde{g}_1(\xi) + \left(\frac{\xi_{1,z}^2}{\rho_1} - \frac{\xi_{2,z}^2}{\rho_2}\right)\tilde{g}_2(\xi)\right.$$

$$\left. + \vec{\xi}_\perp \bullet (\vec{\xi}_\perp - \vec{k}_{0,\perp})\left(\frac{i(\xi_{1,z}+\xi_{2,z})}{\rho_1\rho_2}\tilde{g}_1(\xi) + \left(\frac{1}{\rho_1} - \frac{1}{\rho_2}\right)\tilde{g}_2(\xi)\right)\right]$$

$$-i\frac{1}{2}\left(\frac{k^3{}_{10,z}}{\rho_1}(1-R_0) - \frac{k^3{}_{20,z}}{\rho_2}(1+R_0)\right) \tag{13.63}$$

The generalized reflection (R) and transmission (T) coefficients are defined by the following equations:

$$< p_1 >= (1+R)p_{inc}(z=0) \tag{13.64}$$

$$< p_2 >= Tp_{inc}(z=0) \tag{13.65}$$

$$\frac{1}{\rho_1}\frac{\partial < p_1 >}{\partial z} = -i\frac{k_{1,z}}{\rho_1}(1-R)p_{inc}(z=0) \tag{13.66}$$

$$\frac{1}{\rho_2}\frac{\partial < p_2 >}{\partial z} = -i\frac{k_{2,z}}{\rho_2}Tp_{inc}(z=0) \tag{13.67}$$

Substituting the above expressions into the boundary conditions:

$$< p_1 > - < p_2 >= F_{10} =< \alpha^2 > \hat{F}_1 p_{inc}(z=0) \tag{13.68}$$

$$\frac{1}{\rho_1}\frac{\partial < p_1 >}{\partial z} - \frac{1}{\rho_2}\frac{\partial < p_2 >}{\partial z} = F_{20} =< \alpha^2 > \hat{F}_2 p_{inc}(z=0) \tag{13.69}$$

one arrives at the following equations for the generalized reflection and transmission coefficients:

$$(1+R) - T =< \alpha^2 > \hat{F}_1 \tag{13.70}$$

$$-i\frac{k_{1,z}}{\rho_1}(1-R) + i\frac{k_{2,z}}{\rho_2}T =< \alpha^2 > \hat{F}_2 \tag{13.71}$$

Solving linear Equations 13.70 and 13.71, one arrives at the following expressions for the generalized reflection and transmission coefficients for the coherent component:

$$R = \frac{k_{1,z}/\rho_1 - k_{2,z}/\rho_2}{k_{1,z}/\rho_1 + k_{2,z}/\rho_2} + <\alpha^2> \left[\frac{k_{2,z}}{\rho_2}\hat{F}_1 - i\hat{F}_2 \right] / (k_{1,z}/\rho_1 + k_{2,z}/\rho_2) \tag{13.72}$$

$$T = \frac{2k_{1,z}/\rho_1}{k_{1,z}/\rho_1 + k_{2,z}/\rho_2} - <\alpha^2> \left[\frac{k_{1,z}}{\rho_1}\hat{F}_1 + i\hat{F}_2 \right] / (k_{1,z}/\rho_1 + k_{2,z}/\rho_2) \tag{13.73}$$

The Fourier transform of the cross correlation function of the scattered field is given by the following expectation value:

$$<\tilde{w}_1(\xi)\tilde{w}_1^*(\xi')> = 2\pi\delta(\xi_\perp - \xi_\perp') <\alpha^2> S(\xi) \left\| \frac{i\xi_{2,z}\tilde{g}_1(\xi)/\rho_2 + \tilde{g}_2(\xi)}{\xi_{1,z}/\rho_1 + \xi_{2,z}/\rho_2} \right\|^2 \tag{13.74}$$

The cross correlation function of the scattered field is given by the following expression:

$$<w_1(r)w_1^*(r')> =$$

$$\frac{1}{2\pi}\int d^2\xi_\perp <\alpha^2> S(\xi) \left\| \frac{i\xi_{2,z}\tilde{g}_1(\xi)/\rho_2 + \tilde{g}_2(\xi)}{\xi_{1,z}/\rho_1 + \xi_{2,z}/\rho_2} \right\|^2 \exp[i\vec{\xi}_\perp \bullet (\vec{r} - \vec{r}') + i\xi_z(z - z')] \tag{13.75}$$

Similarly, the cross correlation of the field scattered into Region Two is given by the following expression.

$$<w_2(r)w_2^*(r')> =$$

$$\frac{1}{2\pi}\int d^2\xi_\perp <\alpha^2> S(\xi) \left\| \frac{-i\xi_{1,z}\tilde{g}_1(\xi)/\rho_1 + \tilde{g}_2(\xi)}{\xi_{1,z}/\rho_1 + \xi_{2,z}/\rho_2} \right\|^2 \exp[i\vec{\xi}_\perp \bullet (\vec{r} - \vec{r}') - i\xi_z(z - z')] \tag{13.76}$$

Suppose the spectrum $S(\xi_\perp)$ is sharply peaked around the origin, so that one is justified by approximating the spectrum as the delta function:

$$S(\xi_\perp) \approx 2\pi\delta^2(\xi_\perp) \tag{13.77}$$

in the above integrals. In this case one arrives at the following asymptotic expression for the coherent reflection coefficient and transmission coefficient, where

$$k_{10,z} = i\sqrt{\vec{k}_{0,\perp} \bullet \vec{k}_{0,\perp} - k_1^2} \tag{13.78}$$

and

$$k_{20,z} = i\sqrt{\vec{k}_{0,\perp} \bullet \vec{k}_{0,\perp} - k_2^2} \tag{13.79}$$

are the vertical wavenumber in Media One and Two in the case of a flat interface, respectively.

$$R = \frac{k_{10,z}/\rho_1 - k_{20,z}/\rho_2}{k_{10,z}/\rho_1 + k_{20,z}/\rho_2} + <\alpha^2> \left[\frac{k_{20,z}}{\rho_2}\hat{F}_{10} - i\hat{F}_{20}\right] / (k_{10,z}/\rho_1 + k_{20,z}/\rho_2) \tag{13.80}$$

$$T = \frac{2k_{10,z}/\rho_1}{k_{10,z}/\rho_1 + k_{20,z}/\rho_2} - <\alpha^2> \left[\frac{k_{10,z}}{\rho_1}\hat{F}_{10} + i\hat{F}_{20}\right] / (k_{10,z}/\rho_1 + k_{20,z}/\rho_2) \tag{13.81}$$

$$\hat{F}_{10} = \frac{1}{ik_{10,z}/\rho_1 + ik_{20,z}/\rho_2}\left(k_{10,z}k_{20,z}\tilde{g}_{10}\left(\frac{1}{\rho_2} - \frac{1}{\rho_1}\right) - i(k_{10,z} + k_{20,z})\tilde{g}_{20}\right)$$
$$+\frac{1}{2}(1+R_0)(k_{10,z}^2 - k_{20,z}^2) \tag{13.82}$$

$$\hat{F}_{20} = \frac{1}{ik_{10,z}/\rho_1 + ik_{20,z}/\rho_2}\left(ik_{10,z}k_{20,z}\tilde{g}_{10}(k_{10,z} + k_{20,z})\frac{1}{\rho_1\rho_2} + \left(\frac{k_{10,z}^2}{\rho_1} - \frac{k_{20,z}^2}{\rho_2}\right)\tilde{g}_{20}\right) \tag{13.83}$$
$$-\frac{i}{2}\{(1-R_0)\frac{k_{10,z}^3}{\rho_1} - (1+R_0)\frac{k_{20,z}^3}{\rho_2}\}$$

$$\tilde{g}_{10} = ik_{10,z}(1-R_0) - ik_{20,z}(1+R_0) \tag{13.84}$$

$$\tilde{g}_{20} = (1+R_0)\{\frac{k_{10,z}^2}{\rho_1} - \frac{k_{20,z}^2}{\rho_2}\} \tag{13.85}$$

$$R_0 = \frac{k_{10,z}/\rho_1 - k_{20,z}/\rho_2}{k_{10,z}/\rho_1 + k_{20,z}/\rho_2} \tag{13.86}$$

After some tedious algebra, the following simplifications maybe made to the above equations:

$$R = R_0(1 - 2k_{10,z}^2 <\alpha^2>) \tag{13.87}$$

$$T = T_0(1 + \frac{1}{2} <\alpha^2> (k_{10,z} - k_{20,z})^2) \tag{13.88}$$

The reflection and transmission coefficients thus given agree with Equations 57 and 58 in Reference 18. Equation 13.87 also agrees with the low frequency limit of the Kirchhoff approximation, which gives an effective reflections coefficient for the mean field of the following form:

$$R = R_0 \exp[-2k_{10,z}^2 < \alpha^2 >] \tag{13.89}$$

Having derived the effects of a surface with small wave-height surface roughness on the reflected and transmitted field. Let us reformulate this result into a format amenable to calculation in a waveguide.

Let p_{10} and p_{20} represent the solution for the pressure field in Media One and Two in the absence of surface roughness. They satisfy the boundary conditions:

$$p_{10} = p_{20} \tag{13.90}$$

$$\frac{1}{\rho_1}\frac{dp_{10}}{dz} = \frac{1}{\rho_2}\frac{dp_{20}}{dz} \tag{13.91}$$

at the interface between Media One and Two, and the differential equations:

$$\frac{d^2 p_{10}}{dz^2} + (k_1^2 - q^2) p_{10} = 0 \tag{13.92}$$

$$\frac{d^2 p_{20}}{dz^2} + (k_2^2 - q^2) p_{20} = 0 \tag{13.93}$$

Let δp_1 and δp_2 denote the difference

$$\delta p_1 = < p_1 > - p_{10} \tag{13.94}$$

$$\delta p_2 = < p_2 > - p_{20} \tag{13.95}$$

of the mean field in Media One and Two in the presence of surface roughness and the flat surface field in Media One and Two. They satisfy the boundary conditions

$$\delta p_1 - \delta p_2 = F_1 = < \alpha^2 > (b_1 p_{10} + b_2 \frac{1}{\rho_1}\frac{dp_{10}}{dz}) = < \alpha^2 > (b_1 p_{20} + b_2 \frac{1}{\rho_1}\frac{dp_{20}}{dz}) \tag{13.96}$$

$$\frac{1}{\rho_1}\frac{d}{dz}\delta p_1 - \frac{1}{\rho_2}\frac{d}{dz}\delta p_2 = F_2 = < \alpha^2 > (c_1 p_{10} + c_2 \frac{1}{\rho_1}\frac{dp_{10}}{dz}) = < \alpha^2 > (c_1 p_{20} + c_2 \frac{1}{\rho_1}\frac{dp_{20}}{dz}) \tag{13.97}$$

at the interface between Media One and Two, and satisfy the same differential equations as the functions p_{10} and p_{20}, respectively. The source terms F_1 and F_2 are given by the following expression:

$$F_1 = \frac{<\alpha^2>}{2\pi}\int d^2\xi_\perp \frac{S(\xi_\perp - k_{0,\perp})}{i\xi_{1,z}/\rho_1 + i\xi_{2,z}/\rho_2}\left[\xi_{1,z}\xi_{2,z}\tilde{g}_1(\xi)(\frac{1}{\rho_2}-\frac{1}{\rho_1}) - i(\xi_{1,z}+\xi_{2,z})\tilde{g}_2(\xi)\right]$$

$$+\frac{<\alpha^2>}{2}(k_{1,z}^2 - k_{2,z}^2)p_{10} \tag{13.98}$$

$$F_2 = \frac{<\alpha^2>}{2\pi}\int d^2\xi_\perp \frac{S(\xi_\perp - k_{0,\perp})}{i\xi_{1,z}/\rho_1 + i\xi_{2,z}/\rho_2}\times$$

$$\left[\frac{i\xi_{1,z}\xi_{2,z}}{\rho_1\rho_2}(\xi_{1,z}+\xi_{2,z})\tilde{g}_1(\xi) + \left(\frac{\xi_{1,z}^2}{\rho_1}-\frac{\xi_{2,z}^2}{\rho_2}\right)\tilde{g}_2(\xi)\right.$$

$$\left.+\vec{\xi}_\perp \bullet (\vec{\xi}_\perp - \vec{k}_{0,\perp})\left(\frac{i(\xi_{1,z}+\xi_{2,z})}{\rho_1\rho_2}\tilde{g}_1(\xi) + \left(\frac{1}{\rho_1}-\frac{1}{\rho_2}\right)\tilde{g}_2(\xi)\right)\right]$$

$$+\frac{<\alpha^2>}{2}\left(k^2{}_{10,z} - k^2{}_{20,z}\right)\frac{1}{\rho_1}\frac{dp_{10}}{dz} \tag{13.99}$$

where one has neglected higher order terms. The functions $\tilde{g}_1$ and $\tilde{g}_2$ are given by the following expressions in terms of the functions p_{10} and p_{20}:

$$\tilde{g}_1(\xi) = -(\rho_1-\rho_2)\frac{1}{\rho_1}\frac{dp_{10}}{dz} = -(\rho_1-\rho_2)\frac{1}{\rho_2}\frac{dp_{20}}{dz} \tag{13.100}$$

$$\tilde{g}_2(\xi) = (\frac{k_{10,z}^2}{\rho_1} - \frac{k_{20,z}^2}{\rho_2})p_{10} - \vec{k}_\perp \bullet (\vec{\xi}_\perp - \vec{k}_\perp)(\frac{1}{\rho_1}-\frac{1}{\rho_2})p_{10} \tag{13.101}$$

Substituting the above expressions into Equations 13.98 and 13.99, one arrives at the following expressions for the coefficients of the boundary conditions for the perturbed fields:

$$b_1 = \frac{1}{2}(k_{1,z}^2 - k_{2,z}^2)$$

$$-\frac{1}{2\pi}\int d^2\xi_\perp \frac{S(\xi_\perp - k_\perp)}{\xi_{1,z}/\rho_1 + \xi_{2,z}/\rho_2}(\xi_{1,z}+\xi_{2,z})\left[(\frac{k_{1,z}^2}{\rho_1} - \frac{k_{2,z}^2}{\rho_2}) - \vec{k}_\perp \bullet (\vec{\xi}_\perp - \vec{k}_\perp)(\frac{1}{\rho_1}-\frac{1}{\rho_2})\right] \tag{13.102}$$

$$b_2 = \frac{i}{2\pi}\frac{(\rho_1-\rho_2)^2}{\rho_1\rho_2}\int d^2\xi_\perp \frac{S(\xi_\perp - k_{\perp)}}{\xi_{1,z}/\rho_1+\xi_{2,z}/\rho_2}\xi_{1,z}\xi_{2,z} \tag{13.103}$$

$$c_1 = -\frac{i}{2\pi}\int d^2\xi_\perp \frac{S(\xi_\perp - k_\perp)}{\xi_{1,z}/\rho_1+\xi_{2,z}/\rho_2}(\frac{\xi_{1,z}^2}{\rho_1}-\frac{\xi_{2,z}^2}{\rho_2})\{\frac{k_{1,z}^2}{\rho_1}-\frac{k_{2,z}^2}{\rho_2}-\vec{k}_\perp \bullet(\vec{\xi}_\perp-\vec{k}_\perp)(\frac{1}{\rho_1}-\frac{1}{\rho_2})\}$$

$$-\frac{i}{2\pi}(\frac{1}{\rho_1}-\frac{1}{\rho_2})\int d^2\xi_\perp \frac{S(\xi_\perp - k_\perp)}{\xi_{1,z}/\rho_1+\xi_{2,z}/\rho_2}\vec{\xi}_\perp \bullet(\vec{\xi}_\perp-\vec{k}_\perp)\{\frac{k_{1,z}^2}{\rho_1}-\frac{k_{2,z}^2}{\rho_2}-\vec{k}_\perp \bullet(\vec{\xi}_\perp-\vec{k}_\perp)(\frac{1}{\rho_1}-\frac{1}{\rho_2})\} \tag{13.104}$$

$$c_2 = \frac{1}{2}(k_{1,z}^2-k_{2,z}^2)-\frac{1}{2\pi}\frac{(\rho_1-\rho_2)}{\rho_1\rho_2}\int d^2\xi_\perp \frac{S(\xi_\perp - k_\perp)}{\xi_{1,z}/\rho_1+\xi_{2,z}/\rho_2}(\xi_{1,z}+\xi_{2,z})\{\xi_{1,z}\xi_{2,z}+\vec{\xi}_\perp \bullet(\vec{\xi}_\perp-\vec{k}_\perp)\} \tag{13.105}$$

Let us make the simplifying assumption, that the surface roughness spectrum is isotropic. In this case one can reduce the above expressions into the following relations:

$$b_1 = \frac{1}{2}(k_1^2-k_2^2)-(\frac{k_1^2}{\rho_1}-\frac{k_2^2}{\rho_2})\int_0^\infty q'dq'\frac{S(q'-q)}{\xi_{1,z}/\rho_1+\xi_{2,z}/\rho_2}(\xi_{1,z}+\xi_{2,z}) \tag{13.106}$$

$$b_2 = +i\frac{(\rho_1-\rho_2)^2}{\rho_1\rho_2}\int_0^\infty q'dq'\frac{S(q'-q)}{\xi_{1,z}/\rho_1+\xi_{2,z}/\rho_2}\xi_{1,z}\xi_{2,z} \tag{13.107}$$

$$c_1 = -i(\frac{k_1^2}{\rho_1}-\frac{k_2^2}{\rho_2})^2\int_0^\infty q'dq'\frac{S(q'-q)}{\xi_{1,z}/\rho_1+\xi_{2,z}/\rho_2} \tag{13.108}$$

$$c_2 = \frac{1}{2}(k_1^2-k_2^2)-\frac{(\rho_1-\rho_2)}{\rho_1\rho_2}\int_0^\infty q'dq'\frac{S(q'-q)}{\xi_{1,z}/\rho_1+\xi_{2,z}/\rho_2}(\xi_{1,z}+\xi_{2,z})\{\xi_{1,z}\xi_{2,z}+q'^2\} \tag{13.109}$$

Here $q = |\vec{k}_\perp|$ is the magnitude of the horizontal wavenumber. The above expressions can be substituted into Equations 13.96 and 13.97 for the boundary conditions of the perturbed field.

14. EFFECT OF SMALL WAVE-HEIGHT SURFACE ROUGHNESS ON PROPAGATION OF SOUND IN THE OCEAN

This section describes the effects of surface and bottom roughness on propagation of sound in an otherwise horizontally stratified waveguide.

Let

$$z = \alpha_T(x, y) \tag{14.1}$$

and

$$z = d + \alpha_B(x, y) \tag{14.2}$$

denote a realization of the randomly rough surface and bottom of the waveguide, respectively. The random fields α_T and α_B are homogeneous, zero mean random fields with cross correlation functions of the form:

$$<\alpha_T(r)\alpha_T(r')>=<\alpha_T{}^2>\sigma_T(r-r')=\frac{<\alpha_T{}^2>}{2\pi}\int d^2\xi_\perp S_T(\xi_\perp)\exp[+i\vec{\xi}_\perp\bullet(\vec{r}-\vec{r}')] \tag{14.3}$$

$$<\alpha_B(r)\alpha_B(r')>=<\alpha_B{}^2>\sigma_B(r-r')=\frac{<\alpha_B{}^2>}{2\pi}\int d^2\xi_\perp S_B(\xi_\perp)\exp[+i\vec{\xi}_\perp\bullet(\vec{r}-\vec{r}')] \tag{14.4}$$

where $S_T(\xi_\perp)$ and $S_B(\xi_\perp)$ are the spectral density of the rms wave height of the surface and bottom, respectively.

In the case of a flat surface and bottom, the normal mode contribution to the propagation of sound in the waveguide is of the following form:

$$p_{\mathrm{mode}} = \frac{i}{4\rho(z_s)}\sum_n H_0^{(1)}(q_n r)F(z:q_n)F(z_s:q_n) \tag{14.5}$$

where $F(z:q_n)$ is the depth function of the n'th normal mode. The depth function is a solution of the differential equation:

$$\rho(z)\frac{d}{dz}\frac{1}{\rho(z)}\frac{d}{dz}F(z:q_n)+(k(z)^2-q_n^2)F(z:q_n)=0 \tag{14.6}$$

subject to the following boundary conditions:

$$f_T(q_n)F(z=0:q_n)+\frac{g_T(q_n)}{\rho(0)}\frac{dF(z=0:q_n)}{dz}=0 \tag{14.7}$$

$$f_B(q_n)F(z=d:q_n)+\frac{g_B(q_n)}{\rho(d)}\frac{dF(z=d:q_n)}{dz}=0 \tag{14.8}$$

$$f_T(q)=\frac{ih(0:q)}{\rho(0)}(1-R_{T,0}(q)) \tag{14.9}$$

$$g_T(q)=(1+R_{T,0}(q)) \tag{14.10}$$

$$f_B(q)=\frac{-ih(d:q)}{\rho(d)}(1-R_{B,0}(q)) \tag{14.11}$$

$$g_B(q)=(1+R_{B,0}(q)) \tag{14.12}$$

Here,

$$h(z:q)=+i\sqrt{q^2-k^2(z)} \tag{14.13}$$

is the vertical wavenumber, and

$$R_{T,0}(q)=-1 \tag{14.14}$$

$$R_{B,0}(q)=\frac{h(d:q)/\rho(d)-h_B(d:q)/\rho_B(d)}{h(d:q)/\rho(d)+h_B(d:q)/\rho_B(d)} \tag{14.15}$$

are the reflection coefficients of the surface and bottom in the case of a flat surface and bottom.

Assume the sediment is modeled as a fluid layer, where

$$h_B(z:q)=+i\sqrt{q^2-{k_B}^2(z)} \tag{14.16}$$

is the vertical wavenumber in the sediment, $k_B(z)$ is the wavenumber in the sediment, and $\rho_B(z)$ is the density of the sediment.

The effect of small wave-height surface roughness on the coherent (mean) component of the normal modes can be accounted for by substituting Equation 13.80 for the effective reflection coefficient of the rough interface into Equation 14.9 through 14.12. In evaluating the parameters the quantities in Section 13 are defined by the following limiting value of the parameters on the two sides of the interface at z=d:

$$\rho_1 = \lim_{\varepsilon \to 0} \rho(z = d + \varepsilon) \tag{14.17}$$

$$\rho_2 = \lim_{\varepsilon \to 0} \rho(z = d - \varepsilon) \tag{14.18}$$

$$k_1 = \lim_{\varepsilon \to 0} k(z = d + \varepsilon) \tag{14.19}$$

$$k_2 = \lim_{\varepsilon \to 0} k(z = d - \varepsilon) \tag{14.20}$$

An alternative method of accounting for small wave-height roughness is to treat the effects of surface roughness as a perturbation of the normal mode equations. Let $F(z:q_n)$ denote the depth functions of the unperturbed waveguide with flat boundaries.

These functions are solutions of the differential equation:

$$\frac{d^2F(z:q)}{dz^2} + (k^2(z) - q^2)F(z:q) = 0 \tag{14.21}$$

and satisfy the following normalization condition.

$$\int_0^{+\infty} dz F(z:q)F(z:q)/\rho(z) = 1 \tag{14.22}$$

Here, one must make the assumption the density is piece-wise constant. The effect of surface roughness is to perturb the above eigenvalues and eigenfunctions.

Let $F(z:q_n) + \delta F(z:q_n)$ denote the perturbed depth function and $q_n + \delta q_n$ the perturbed eigenvalue due to roughness of the interface at z=d. The perturbed eigenfunction satisfies the following boundary condition at z=d:

$$\lim_{\varepsilon \to 0}(\delta F(d+\varepsilon:q) - \delta F(d-\varepsilon:q)) = <\alpha^2>(b_1 F(z=d:q) + b_2 \frac{1}{\rho}\frac{dF(z=d:q)}{dz}) \tag{14.23}$$

$$\begin{aligned}&\lim_{\varepsilon \to 0}(\frac{1}{\rho(d+\varepsilon)}\frac{d}{dz}\delta F(d+\varepsilon:q) - \frac{1}{\rho(d-\varepsilon)}\frac{d}{dz}\delta F(d-\varepsilon:q)) = \\ &<\alpha^2>(c_1 F(z=d:q) + c_2 \frac{1}{\rho}\frac{dF(z=d:q)}{dz})\end{aligned} \tag{14.24}$$

The coefficients on the right hand side of the boundary conditions are given by either Equations 13.102 through 13.105 for a general surface roughness spectrum, or by Equations 13.106 through 13.109 for an isotropic surface roughness spectrum.

The perturbed depth function satisfies the differential equation:

$$\frac{d^2F(z:q)+\delta F(z:q)}{dz^2}+(k^2(z)-q^2-\delta q^2)(F(z:q)+\delta F(z:q))=0 \tag{14.25}$$

where δq^2 is the perturbation of the horizontal wavenumber squared. Keeping only first order terms, one arrives at the differential equation:

$$\frac{d^2\delta F(z:q)}{dz^2}+(k^2(z)-q^2)\delta F(z:q)=\delta q^2F(z:q) \tag{14.26}$$

relating the perturbed depth functions and the perturbed eigenvalue. Here the perturbed eigenvalue acts as a source term for the perturbed depth function. This result is typical of first order perturbation theory.

Combining Equations 14.21 and 14.26 one arrives at the following result for the depth derivative of the Wronskian of the perturbed and unperturbed depth functions:

$$F(z:q)\frac{d\delta F(z:q)}{dz}-\delta F(z:q)\frac{dF(z:q)}{dz}=\delta q^2F(z:q)F(z:q) \tag{14.27}$$

Dividing the above product by the density and integrating the resulting product over the entire waveguide, one arrives at the following relation.

$$\delta q^2=\delta q^2\int_0^\infty dz\frac{F(z:q)F(z:q)}{\rho(z)}=\int_0^\infty dz\frac{F(z:q)\delta F''(z:q)-F''(z:q)\delta F(z:q)}{\rho(z)} \tag{14.28}$$

Here, one has made use of the orthogonality condition of the unperturbed depth functions. Assuming the density is piece-wise constant, the above integrand is a total derivative in each layer of the waveguide except at the interface z=d, where the perturbed depth function is discontinuous. In this case, the above integral is equal to the following limit:

$$\delta q^2=-\lim_{\varepsilon\to 0}(F(z:q)\delta F'(z:q)-F'(z:q)\delta F(z:q))/\rho(z)\,|_{z=d-\varepsilon}^{d+\varepsilon} \tag{14.29}$$

Using the boundary conditions given by Equations 14.23 and 14.24 for the discontinuity of the perturbed field at the boundary, one arrives at the following expression for the perturbed eigenvalue:

$$\delta q^2=-<\alpha^2>\{c_1F(d:q)^2+(c_2-b_1)F(d:q)F'(d:q)/\rho-b_2(F'(d:q)/\rho)^2\} \tag{14.30}$$

To first order in perturbation theory, the perturbed depth functions are equal to the unperturbed depth functions evaluated at the perturbed eigenvalue $q+\delta q$. In particular, the perturbed depth function $\delta F(z:q)$ is given by the difference

$$\delta F(z:q) = F(z:q+\delta q) - F(z:q) \tag{14.30}$$

Note, in the above first order perturbation theory, the effect of multiple rough surfaces is additive to first order. Thus, one can represent the effect of a rough surface and bottom by the sum of the perturbations of each surface on the unperturbed eigenvalues.

In the case of a sound soft surface at z = 0, the perturbation of the eigenvalue is given by the following relationship:

$$\delta q^2 = + < \alpha^2 > b_2 (F'(z=0:q)/\rho(0))^2 \tag{14.31}$$

The coefficient b_2 is given by the following limit $\rho_2 \to 0$ of Equation 13.107 in the case of an isotropic surface roughness:

$$b_2 = +i\rho_1 \int_0^\infty q' dq' S(q'-q) \xi_{1,z} \tag{14.32}$$

In the case of a rigid rough surface at z=D, the perturbed eigenvalue is given by the following expression:

$$\delta q^2 = - < \alpha^2 > c_1 (F(z=0:q))^2 \tag{14.33}$$

The coefficient c_1 is given by the following limit $\rho_1 \to +\infty$ of Equation 13.108 in the case of an isotropic rough surface:

$$c_1 = -i\rho_2 \left(\frac{k_2^2}{\rho_2}\right)^2 \int_0^\infty q' dq' \frac{S(q'-q)}{\xi_{2,z}} \tag{14.34}$$

15. ASYMPTOTIC EVALUATION OF THE TIME DOMAIN SOLUTION

This section presents an asymptotic analysis of the propagation of a Gaussian pulse in a waveguide in the time domain. This analysis is based on a saddle point evaluation of the asymptotic evaluation of the normal mode expression for the Green's Function.

The Green's Function for the propagation of a monochromatic signal in a waveguide is of the following form.

$$G(r,z_s,z)=\frac{i}{4\rho(z_s)}\sum_n H_0^{(1)}(q_n r)F(z_s:q_n)F(z:q_n) \tag{15.1}$$

Substituting the asymptotic expression

$$H_0^{(1)}(qr)\approx\sqrt{\frac{2}{\pi qr}}e^{i(qr-\pi/4)} \tag{15.2}$$

for the Hankel Function into Equation 15.1, one obtains the following asymptotic expansion of the Green's Function:

$$G(r,z_s,z)\approx\frac{i}{4\rho(z_s)}\sum_n\sqrt{\frac{2}{\pi q_n r}}e^{+i(q_n r-\pi/4)}F(z_s:q_n)F(z:q_n) \tag{15.3}$$

Propagation of a pulse is described by the Fourier Transform:

$$G(r,z_s,z:t)=\int_{-\infty}^{+\infty}S(\omega)G(r,z_s,z:\omega)e^{-i\omega t}d\omega \tag{15.4}$$

where $S(\omega)$ is the spectrum of the incident signal. In the following one may assume a spectrum of the following form:

$$S(\omega)=\exp(-\frac{(\omega-\omega_0)^2}{2\Delta\omega^2}) \tag{15.5}$$

where ω_0 is the center frequency, and $\Delta\omega$ is the bandwidth of the pulse. The incident signal is of the form

$$s(t)=\int_{-\infty}^{+\infty}S(\omega)e^{-i\omega t}d\omega=\sqrt{2\pi}\Delta\omega\exp(-t^2\Delta\omega^2/2-i\omega_0 t) \tag{15.6}$$

where

$$\tau = \frac{1}{\Delta\omega} \tag{15.7}$$

represents the pulse length of the incident signal.

Substituting Equation 15.3 and 15.5 into Equation 15.4, one obtains the following expression for the propagation of a Gaussian pulse in a waveguide:

$$G(r,z_s,z:t) =$$
$$\frac{i}{4\rho(z_s)}\sum_n \int_{-\infty}^{+\infty} \exp(i(q_n r - \omega t - \pi/4) - (\omega-\omega_0)^2/2\Delta\omega^2) F(z_s : q_n) F(z : q_n) d\omega \tag{15.8}$$

One must now assume that this integral is dominated by the exponential term and the depth functions inside the integral are approximately equal to their value at the center frequency of the pulse. Under this assumption, evaluation of Equation 15.8 reduces to the evaluation of integrals of the following form:

$$I_n(r,z_s,z:t) = \int_{-\infty}^{+\infty} \exp(i(q_n r - \omega t - \pi/4) - (\omega-\omega_0)^2/2\Delta\omega^2) d\omega \tag{15.9}$$

where the horizontal wavenumber $q_n(\omega)$ is an implicit function of frequency. Expanding the argument of the exponential function about the center frequency, one obtains the following expression for the integral:

$$I_n(r,z_s,z:t) \approx \exp(i(q_n(\omega_0)r - \omega_0 t - \pi/4) \int_{-\infty}^{+\infty} \exp(-\alpha^2(\omega-\omega_0)^2 - i(\omega-\omega_0)(t-t_n)) d\omega \tag{15.10}$$

where the reciprocal of the group velocity of the mode is:

$$q'_n(\omega) = \frac{dq_n(\omega)}{d\omega} \tag{15.11}$$

the rate of change of the reciprocal of the group velocity is

$$q''_n(\omega) = \frac{d^2 q_n(\omega)}{d\omega^2} \tag{15.12}$$

the travel time for the mode is

$$t_n = rq'_n(\omega_0) \tag{15.13}$$

and the complex width squared of the effective Gaussian is

$$\alpha^2 = \frac{1}{2\Delta\omega^2} - \frac{i}{2} q''_n(\omega_0) \tag{15.14}$$

Equation 15.10 can now be computed by completing the square in the argument of the exponential function to recast the integral into a simple Gaussian of the form:

$$\frac{\sqrt{\pi}}{\alpha} = \int_{-\infty}^{+\infty} \exp(-\alpha^2 x^2)dx \tag{15.15}$$

The resulting integral is given by the following expression:

$$I_n(r, z_s, z : t) \approx \frac{\sqrt{\pi}}{\alpha} \exp(i(q_n(\omega_0)r - \omega_0 t - \pi/4) - (t - t_n)^2 / 4\alpha^2) \tag{15.16}$$

Substituting Equation 15.16 into Equation 15.8, one obtains the following asymptotic expression for the time domain representation of a Gaussian pulse propagating in a waveguide:

$$G(r, z_s, z : t) =$$

$$\frac{i}{4\rho(z_s)} \sum_n \sqrt{\frac{2}{q_n r \pi}} \frac{\sqrt{\pi}}{\alpha} F(z_s : q_n) F(z : q_n) \exp(i(q_n r - \omega_0 t - \pi/4) - (t - t_n)^2 / 4\alpha^2) \tag{15.17}$$

where the normal modes are evaluated at the center frequency.

16. REPRESENTATIONS OF CYLINDRICAL WAVE FUNCTIONS

This section focuses on different representations of cylindrical wave functions.

The regular cylindrical wave functions are given by the following expressions:

$$\operatorname{Re}\chi_{\sigma m}^{(+)}(r:\alpha)=\sqrt{\frac{\varepsilon(m)}{8\pi}}J_m(k\rho\sin(\alpha))e^{+ikz\cos(\alpha)}\begin{Bmatrix}\cos(m\varphi)\sigma=e\\ \sin(m\varphi)\sigma=o\end{Bmatrix} \tag{16.1}$$

$$\operatorname{Re}\chi_{\sigma m}^{(-)}(r:\alpha)=\sqrt{\frac{\varepsilon(m)}{8\pi}}J_m(k\rho\sin(\alpha))e^{-ikz\cos(\alpha)}\begin{Bmatrix}\cos(m\varphi)\sigma=e\\ \sin(m\varphi)\sigma=o\end{Bmatrix} \tag{16.2}$$

Here, the function $\varepsilon(m)$ is defined by the following relationship:

$$\varepsilon(m)=\begin{Bmatrix}1,m=0\\ 2,m\neq 0\end{Bmatrix} \tag{16.3}$$

The irregular cylindrical wave functions corresponding to outgoing waves are given by the following expressions:

$$\chi_{\sigma m}^{(+)}(r:\alpha)=\sqrt{\frac{\varepsilon(m)}{8\pi}}H^{(1)}{}_m(k\rho\sin(\alpha))e^{+ikz\cos(\alpha)}\begin{Bmatrix}\cos(m\varphi)\sigma=e\\ \sin(m\varphi)\sigma=o\end{Bmatrix} \tag{16.4}$$

$$\chi_{\sigma m}^{(-)}(r:\alpha)=\sqrt{\frac{\varepsilon(m)}{8\pi}}H^{(1)}{}_m(k\rho\sin(\alpha))e^{-ikz\cos(\alpha)}\begin{Bmatrix}\cos(m\varphi)\sigma=e\\ \sin(m\varphi)\sigma=o\end{Bmatrix} \tag{16.5}$$

The variable α represents the angle of the wave with respect to the z-axis. The horizontal wavenumber q, and vertical wavenumber h are given by the following expressions in terms of this angle:

$$q=k\sin(\alpha) \tag{16.6}$$

$$h=k\cos(\alpha) \tag{16.7}$$

The downward-going cylindrical wave functions and upward-going wave functions are related by the following relationship:

$$\operatorname{Re}\chi_{\sigma m}^{(-)}(r:\alpha)=\operatorname{Re}\chi_{\sigma m}^{(+)}(r:\pi-\alpha) \tag{16.8}$$

The following expansion

$$\exp[ix\cos(\alpha)] = \sum_{n=-\infty}^{+\infty} i^n J_n(x) e^{+in\alpha} = J_0(x) + 2\sum_{n=1}^{+\infty} i^n J_n(x)\cos(n\alpha) \tag{16.9}$$

follows by substituting the expression

$$t = i\exp[i\alpha] \tag{16.10}$$

into the generating function for the regular Bessel Functions:

$$\exp[x(t - t^{-1})/2] = \sum_{n=-\infty}^{+\infty} J_n(x) t^n \tag{16.11}$$

Equation 16.8 serves as the basis for the following expansion of a plane wave in terms of the regular cylindrical wave functions:

$$\begin{aligned} \exp[\pm i\vec{k} \bullet \vec{r}] &= \sum_{m=0}^{+\infty} \varepsilon(m)(\pm i)^m J_m(k\rho\sin(\alpha))\exp[\pm ikz\cos(\alpha)]\cos(m(\varphi - \beta)) \\ &= 4\pi \sum_{m=0}^{+\infty}\sum_{\sigma} (\pm i)^m \operatorname{Re}\chi_{\sigma m}^{(\pm)}(r:\alpha)\sqrt{\frac{\varepsilon(m)}{2\pi}}\begin{Bmatrix} \cos(m\beta), \sigma = e \\ \sin(m\beta), \sigma = o \end{Bmatrix} \end{aligned} \tag{16.12}$$

where the components of the wave vector are given by the following expressions:

$$k_x = k\sin(\alpha)\cos(\beta) \tag{16.13}$$

$$k_y = k\sin(\alpha)\sin(\beta) \tag{16.14}$$

$$k_z = k\cos(\alpha) \tag{16.15}$$

Using the orthogonality of the trigonometric functions and the above expansion of the plane wave in terms of regular cylindrical wave functions, one arrives at the following plane wave expansion of the regular cylindrical wave functions:

$$\operatorname{Re}\chi_{\sigma m}^{(\pm)}(r:\alpha) = \frac{1}{4\pi(\pm i)^m}\int_0^{2\pi}\sqrt{\frac{\varepsilon(m)}{2\pi}}\begin{Bmatrix} \cos(m\beta), \sigma = e \\ \sin(m\beta), \sigma = o \end{Bmatrix}\exp[\pm i\vec{k} \bullet \vec{r}]d\beta \tag{16.16}$$

The above integral representation of the regular cylindrical functions may also be inferred from the following integral representation of the regular cylindrical Bessel Function:

$$J_n(x) = \frac{1}{\pi(\pm i)^n}\int_0^{\pi}\exp[\pm ix\cos(\beta)]\cos(n\beta)d\beta = \frac{1}{2\pi(\pm i)^n}\int_0^{2\pi}\exp[\pm ix\cos(\beta)]\cos(n\beta)d\beta$$
$$= \frac{1}{2\pi(\pm i)^n}\int_0^{2\pi}\exp[\pm ix\cos(\beta) \pm in\beta]d\beta \tag{16.17}$$

Making the following substitution:

$$\vec{k}\bullet\vec{r} = k\rho\sin(\alpha)\cos(\beta-\varphi) + kz\cos(\alpha) \tag{16.18}$$

of the dot product in Equation 16.16, one arrives at the following integral:

$$\operatorname{Re}\chi_{\sigma m}^{(\pm)}(r:\alpha) =$$
$$\frac{1}{4\pi(\pm i)^m}\int_0^{2\pi}\sqrt{\frac{\varepsilon(m)}{2\pi}}\begin{Bmatrix}\cos(m\beta), \sigma = e\\ \sin(m\beta), \sigma = o\end{Bmatrix}\exp[\pm i(k\rho\sin(\alpha)\cos(\beta-\varphi) + kz\cos(\alpha)]d\beta \tag{16.19}$$

which is of the form of Equation 16.17, and leads to the result listed in Equation 16.16.

A similar plane wave expansion of the outgoing cylindrical waves:

$$\chi_{\sigma m}^{+}(r:\alpha) = \frac{1}{2\pi i^m}\int_{\pi/2-i\infty}^{-\pi/2+i\infty}\exp[i\vec{k}\bullet\vec{r}]\sqrt{\frac{\varepsilon(m)}{2\pi}}\begin{Bmatrix}\cos(m\beta), \sigma = e\\ \sin(m\beta), \sigma = o\end{Bmatrix}d\beta \tag{16.20}$$

can be obtained from the following integral representation of the Hankel Functions:

$$H_n^{(1)}(x) = \frac{1}{\pi i^n}\int_{\pi/2-i\infty}^{\pi/2+i\infty}\exp[+ix\cos(\beta) - in\beta]d\beta \tag{16.21}$$

The expansions:

$$G(r,r') = \exp[ik\,|r-r'|]/4\pi\,|r-r'| = 2ik\sum_{\sigma m}\int_{C_\pm}\sin(\alpha)d\alpha\operatorname{Re}\chi_{\sigma m}^{+}(r:\alpha)\operatorname{Re}\chi_{\sigma m}^{-}(r':\alpha) \tag{16.22}$$

$$G(r,r') = \exp[ik\,|r-r'|]/4\pi\,|r-r'| = 2ik\sum_{\sigma m}\int_{C}\sin(\alpha)d\alpha\chi_{\sigma m}^{\pm}(r_>:\alpha)\operatorname{Re}\chi_{\sigma m}^{\mp}(r_<:\alpha) \tag{16.23}$$

of the free field Green's Function follow from the integral representation of the free field Green's Function. The contours C and $C_\pm$ are the following contours:

$$C_+ = [0, \pi/2] \cup [\pi/2, \pi/2 - i\infty) \tag{16.24}$$

$$C_- = (\pi/2 + i\infty, \pi/2] \cup [\pi/2, \pi] \tag{16.25}$$

$$C = (+i\infty, 0] \cup [0, \pi] \cup [\pi, \pi - i\infty) \tag{16.26}$$

The contour C_+ is used in the case $(z - z') > 0$ and the contour C_- is used in the case $(z - z') < 0$ in Equation 16.22. Conversely, the contour C_- is used in the case $(z - z') > 0$, and the contour C_+ is used in the case $(z - z') < 0$ in Equation 16.23.

A derivation of Equations 16.22 and 16.23 follows. Start with the following integral representation of the free field Green's Function:

$$G(r, r') = \int \frac{d^3 p}{(2\pi)^3} \exp[+i\vec{p} \bullet (\vec{r} - \vec{r}')] / (p^2 - k^2 - i0) \tag{16.27}$$

Introduce the variable

$$M = \sqrt{p_z^2 - k^2}$$

and express the denominator of Equation 16.27 as the following product:

$$(p^2 - k^2 - i0) = (p_y + i\sqrt{p_x^2 + M^2} - i0)(p_y - i\sqrt{p_x^2 + M^2} - i0) \tag{16.28}$$

Equation 16.27 takes on the following form:

$$G(r, r') =$$

$$\frac{1}{(2\pi)^3} \int_{-\infty}^{+\infty} dp_z \exp[+ip_z z''] \int_{-\infty}^{+\infty} dp_x \exp[+ip_x x''] \int_{-\infty}^{+\infty} dp_y \exp[+ip_y y''] / (p_y + i\sqrt{p_x^2 + M^2} - i0)(p_y - i\sqrt{p_x^2 + M^2} - i0) \tag{16.29}$$

The integral over p_y can be performed by the method of residues, where one either closes the contour in the upper half or lower half plane, depending upon whether the variable $y'' = (y - y')$ is positive or negative. Performing the integration over the variable p_y one obtains the following expression for Equation 16.29:

$$G(r, r') =$$

$$\frac{\pm 2\pi i}{\pm 2i} \frac{1}{(2\pi)^3} \int_{-\infty}^{+\infty} dp_z \exp[+ip_z z''] \int_{-\infty}^{+\infty} dp_x \exp[+ip_x x''] \exp[\mp y'' \sqrt{p_x^2 + M^2}] / \sqrt{p_x^2 + M^2} \tag{16.30}$$

where the upper sign is chosen in the case $y''=(y-y')>0$ is positive, and the lower sign is chosen in the case it is negative. Introducing the following transformation:

$$p_x = iM\sin(\beta) \tag{16.31}$$

in Equation 16.30, one arrives at the following contour integral:

$$G(r,r') = \frac{i}{8\pi^2}\int_{-\infty}^{+\infty} dp_z \exp[+ip_z z''] \int_{\pi/2-i\infty}^{\pi/2+i\infty} d\beta \exp[-x'' M\sin(\beta) \mp y'' M\cos(\beta)] \tag{16.32}$$

Expressing the argument of the exponential in polar coordinates:

$$x'' M\sin(\beta) \pm y'' M\cos(\beta) = M\rho''\cos(\beta \pm \varphi) \tag{16.33}$$

Equation 16.32 takes on the following form:

$$G(r,r') = \frac{i}{8\pi^2}\int_{-\infty}^{+\infty} dp_z \exp[+ip_z z''] \int_{\pi/2-i\infty}^{\pi/2+i\infty} d\beta \exp[-\rho'' M\cos(\beta \pm \varphi)] \tag{16.34}$$

The integral over the variable β is equal to the following expression:

$$\int_{\pi/2-i\infty}^{\pi/2+i\infty} d\beta \exp[-M\rho''\cos(\beta \pm \varphi)] = -i\int_{-\infty}^{+\infty} d\lambda \exp[-M\rho''\cosh(\lambda) = \pi H_0^{(1)}(iM\rho'')] \tag{16.35}$$

Substituting the above expression into Equation 16.34, one arrives at the following expression for the free field Green's Function:

$$G(r,r') = \frac{i}{8\pi}\int_{-\infty}^{+\infty} dp_z \exp[+ip_z z''] H_0^{(1)}(i\sqrt{p_z^2 - k^2}\,\rho'') \tag{16.36}$$

Introducing the following coordinate transformation:

$$p_z = k\cos(\alpha) \tag{16.37}$$

$$iM = i\sqrt{p_z^2 - k^2} = k\sin(\alpha) \tag{16.38}$$

into Equation 16.36, one arrives at the following integral representation of the free field Green's Function.:

$$G(r,r') = \frac{ik}{8\pi}\int_{+i\infty}^{\pi-i\infty} d\alpha \sin(\alpha) \exp[ikz''\cos(\alpha)] H_0^{(1)}(k\rho''\sin(\alpha)) \tag{16.39}$$

Replacing the integral over the complex angle by an integration over the horizontal wavenumber:

$$q = k\sin(\alpha) \tag{16.40}$$

one arrives at the following version of Equation 16.39:

$$G(r,r') = \frac{ik}{8\pi}\int_{+i\infty-0}^{+i\infty+0} \frac{qdq}{h}\exp[+ihz"]H_0^{(1)}(q\rho") \tag{16.41}$$

where the integral is around the branch cut of the vertical wave number:

$$h(q) = i\sqrt{q^2 - k^2} = k\cos(\alpha) \tag{16.42}$$

that is, the integral is along the following contour:

$$(+i\infty - \varepsilon, 0 - i\varepsilon] \bigcup [0 - i\varepsilon, k - i\varepsilon] \bigcup [k + i\varepsilon, 0 + i\varepsilon] \bigcup [0 + i\varepsilon, i\infty + \varepsilon] \tag{16.43}$$

where ε is an infinitesimal displacement.

One can deform the contour around the branch cut of the vertical wave number into one along the real axis, provided one chooses the sign of the vertical wave number such that the imaginary part of the product $\mathrm{Im}(hz") > 0$ is positive. In this case, Equation 16.41 takes on the following form:

$$G(r,r') = \frac{ik}{8\pi}\int_{-\infty-i0}^{+\infty-i0} \frac{qdq}{h}\exp[+ih\,|\,z"|]H_0^{(1)}(q\rho") \tag{16.44}$$

Using the relationships:

$$H_0^{(2)}(x) = H_0^{(1)}(-x) \tag{16.45}$$

$$J_0(x) = \frac{1}{2}(H_0^{(1)}(x) + H_0^{(2)}(x)) \tag{16.46}$$

one can express Equation 16.44 into the following integral over the regular Bessel Function:

$$G(r,r') = \frac{ik}{4\pi}\int_{0-i0}^{+\infty-i0} \frac{qdq}{h}\exp[+ih\,|\,z"|]J_0(q\rho") \tag{16.47}$$

Equations 16.44 and 16.47 serve as a basis for generating an integral representation of the free field Green's Function in terms of the cylindrical wave functions.

Substituting the expansion:

$$H_0^{(1)}(q\rho'') = \sum_{m=0}^{+\infty} \varepsilon(m) H_m^{(1)}(q\rho_>) J_m(q\rho_<) \cos(m(\varphi - \varphi')) \tag{16.48}$$

into Equation 16.44, one arrives at the following expression for the free field Green's Function:

$$G(r,r') = i\sum_{m=0}^{+\infty} \frac{\varepsilon(m)}{8\pi} \cos(m(\varphi - \varphi')) \int_{-\infty}^{+\infty} H_m^{(1)}(q\rho_>) J_m(q\rho_<) \exp[+ih\,|\,z - z'|] q\,dq / h \tag{16.49}$$

Equation 16.49 can be expressed in terms of the outgoing and regular cylindrical wave function:

$$G(r,r') = i\sum_{\sigma m} \int_{-\infty}^{+\infty} \frac{q\,dq}{h} \chi_{\sigma m}^{(\pm)}(r_> : q) \operatorname{Re} \chi_{\sigma m}^{(\mp)}(r_< : q) \tag{16.50}$$

where $\vec{r}_> = \vec{r}$ and $\vec{r}_< = \vec{r}\,'$ if $\rho > \rho'$, and $\vec{r}_> = \vec{r}\,'$ and $\vec{r}_< = \vec{r}$ otherwise. The upper sign is chosen if the quantity $(z_> - z_<) > 0$ is positive, and the lower sign is chosen if it is negative. Converting Equation 16.50 into an integral over the complex angle α:

$$q = k \sin(\alpha) \tag{16.51}$$

one arrives at the following expression for the free field Green's Function:

$$G(r,r') = ik \sum_{\sigma m} \int_C \chi_{\sigma m}^{(\pm)}(r_> : \alpha) \operatorname{Re} \chi_{\sigma m}^{(\mp)}(r_< : \alpha) \sin(\alpha)\, d\alpha \tag{16.52}$$

where C is the contour defined by Equation 16.26.

Substituting the expansion:

$$J_0(q\rho'') = \sum_{m=0}^{+\infty} \varepsilon(m) J_m(q\rho) J_m(q\rho') \cos(m(\varphi - \varphi')) \tag{16.53}$$

into Equation 16.47, one arrives at the following expression:

$$G(r,r') = 2i\sum_{m=0}^{+\infty} \frac{\varepsilon(m)}{8\pi} \cos(m(\varphi - \varphi')) \int_0^{+\infty} J_m(q\rho) J_m(q\rho') \exp[+ih\,|\,z - z'|] q\,dq / h \tag{16.54}$$

Expressing Equation 16.54 in terms of the regular cylindrical wave functions, one arrives at the following expression for the free field Green's Function in terms of the cylindrical wave functions:

$$G(r,r') = 2i\sum_{\sigma m}\int_{0}^{+\infty} \mathrm{Re}\,\chi_{\sigma m}^{\pm}(r:q)\,\mathrm{Re}\,\chi_{\sigma m}^{\mp}(r':q)\frac{qdq}{h} \tag{16.55}$$

The upper sign is used in Equation 16.55 if the quantity $(z-z')>0$ is positive, and the lower sign is used if it is negative. Transforming the integral in Equation 16.55 into one over the complex angle α, one obtains the expression:

$$G(r,r') = 2ik\sum_{\sigma m}\int_{C_{\pm}} \mathrm{Re}\,\chi_{\sigma m}^{(+)}(r:\alpha)\,\mathrm{Re}\,\chi_{\sigma m}^{(-)}(r':\alpha)\sin(\alpha)d\alpha \tag{16.56}$$

where the contour C_+ is used if the quantity $(z-z')>0$ is positive, and the contour C_- is chosen if the quantity is negative.

17. REPRESENTATIONS OF SPHERICAL WAVE FUNCTIONS

This section describes various representations of the spherical wave functions.

The regular and outgoing spherical wave functions are defined by the following expressions:

$$\mathrm{Re}\,\psi_{\sigma ml}(r) = j_l(kr)\sqrt{\frac{\varepsilon(m)(2l+1)}{4\pi}\frac{(l-m)!}{(l+m)!}}P_l^m(\cos(\theta))\begin{Bmatrix}\cos(m\varphi), \sigma = e\\ \sin(m\varphi), \sigma = o\end{Bmatrix} \tag{17.1}$$

$$\psi_{\sigma ml}(r) = h^{(1)}{}_l(kr)\sqrt{\frac{\varepsilon(m)(2l+1)}{4\pi}\frac{(l-m)!}{(l+m)!}}P_l^m(\cos(\theta))\begin{Bmatrix}\cos(m\varphi), \sigma = e\\ \sin(m\varphi), \sigma = o\end{Bmatrix} \tag{17.2}$$

where $j_l(kr)$ and $h_l^{(1)}(kr)$ are the regular spherical Bessel Function and Hankel Function of the first kind, respectively. The functions:

$$P_l^m(x) = (-1)^m(1-x^2)^{m/2}\frac{d^m}{dx^m}P_l(x) = \frac{(-1)^m(1-x^2)^{m/2}}{2^l l!}\frac{d^{m+l}}{dx^{m+l}}(x^2-1)^l \tag{17.3}$$

are the real associated Legendre Functions of the first kind. These functions can be extended to the complex plane in two ways. The first method is to define the function

$$P_l^m(\cos(\theta)) = (-1)^m\sin(\theta)^m\frac{d^m}{dx^m}P_l(x=\cos(\theta)) \tag{17.4}$$

in terms of a complex angle θ. The second method is to define the associated Legendre polynomial as the analytic function

$$\widetilde{P}_l^m(z) = (z^2-1)^{m/2}\frac{d^m}{dx^m}P_l(x) = \frac{(z^2-1)^{m/2}}{2^l l!}\frac{d^{m+l}}{dz^{m+l}}(z^2-1)^l \tag{17.5}$$

of the variable z, where the function $\widetilde{P}_l^m(z)$ is related to the real associated Legendre Functions on the branch cut [-1,+1] by the following expressions:

$$\lim_{\varepsilon\to 0}\widetilde{P}_l^m(x+i\varepsilon) = (-i)^m P_l^m(x) \tag{17.6}$$

$$\lim_{\varepsilon\to 0}\widetilde{P}_l^m(x-i\varepsilon) = (+i)^m P_l^m(x) \tag{17.6}$$

In the following one may use the extension of the spherical wave functions by the extension of the associated Legendre polynomials defined by Equation 17.4 by the notation $\mathrm{Re}\psi_{\sigma ml}$ and $\psi_{\sigma ml}$. Similarly, one may use the notation:

$$\mathrm{Re}\tilde{\psi}_{\sigma ml}(r)=j_l(kr)\sqrt{\frac{\varepsilon(m)(2l+1)}{4\pi}\frac{(l-m)!}{(l+m)!}}\tilde{P}_l^m(\cos(\theta))\begin{Bmatrix}\cos(m\varphi),\sigma=e\\ \sin(m\varphi),\sigma=o\end{Bmatrix} \tag{17.7}$$

$$\tilde{\psi}_{\sigma ml}(r)=h^{(1)}{}_l(kr)\sqrt{\frac{\varepsilon(m)(2l+1)}{4\pi}\frac{(l-m)!}{(l+m)!}}\tilde{P}_l^m(\cos(\theta))\begin{Bmatrix}\cos(m\varphi),\sigma=e\\ \sin(m\varphi),\sigma=o\end{Bmatrix} \tag{17.8}$$

to denote the extension of the spherical wave functions to complex variables defined by Equation 17.5. The two representations are related by the following relationships along the branch cut [-1,+1] of the associated Legendre polynomials:

$$\lim_{\varepsilon\to 0}\tilde{\psi}_{\sigma ml}(x+i\varepsilon)=(-i)^m\psi_{\sigma ml}(x) \tag{17.9}$$

$$\lim_{\varepsilon\to 0}\tilde{\psi}_{\sigma ml}(x-i\varepsilon)=(+i)^m\psi_{\sigma ml}(x) \tag{17.10}$$

The complex associated Legendre polynomials satisfy the following addition theorem:

$$\begin{aligned}P_l(\cos(\gamma))&=\sum_{m=0}^{+l}\varepsilon(m)\frac{(l-m)!}{(l+m)!}P_l^m(\cos(\vartheta))P_l^m(\cos(\vartheta'))\cos(m(\varphi-\varphi'))\\ &=\sum_{m=0}^{+l}\varepsilon(m)(-1)^m\frac{(l-m)!}{(l+m)!}\tilde{P}_l^m(\cos(\vartheta))\tilde{P}_l^m(\cos(\vartheta'))\cos(m(\varphi-\varphi'))\end{aligned} \tag{17.11}$$

The complex angle γ is defined by the following expression:

$$\cos(\gamma)=\cos(\vartheta)\cos(\vartheta')+\sin(\vartheta)\sin(\vartheta')\cos(\varphi-\varphi') \tag{17.12}$$

The associated Legendre polynomials satisfy the following orthogonality relationships:

$$\int_{-1}^{+1}P_l^m(x)P_{l'}^m(x)dx=\frac{2}{2l+1}\frac{(l+m)!}{(l-m)!}\delta_l^{l'} \tag{17.13}$$

$$\int_{-1}^{+1}\tilde{P}_l^m(x\pm i0)\tilde{P}_{l'}^m(x\pm i0)dx=(-1)^m\frac{2}{2l+1}\frac{(l+m)!}{(l-m)!}\delta_l^{l'} \tag{17.14}$$

$$\int_{-1}^{+1}P_l^m(x)P_l^{m'}(x)\frac{dx}{\sqrt{1-x^2}}=\frac{1}{m}\frac{(l+m)!}{(l-m)!}\delta_m^{m'},m\neq 0 \tag{17.15}$$

$$\int_{-1}^{+1} \widetilde{P}_l^m(x \pm i0)\widetilde{P}_l^{m'}(x \pm i0)\frac{dx}{\sqrt{1-x^2}} = \frac{(-1)^m}{m}\frac{(l+m)!}{(l-m)!}\delta_m^{m'}, m \neq 0 \tag{17.16}$$

The spherical harmonics are defined by the following expressions:

$$Y_{\sigma ml}(\vartheta,\varphi) = \sqrt{\frac{\varepsilon(m)(2l+1)}{4\pi}\frac{(l-m)!}{(l+m)!}}P_l^m(\cos(\vartheta))\begin{Bmatrix}\cos(m\varphi), \sigma = e \\ \sin(m\varphi), \sigma = o\end{Bmatrix} \tag{17.17}$$

$$\widetilde{Y}_{\sigma ml}(\vartheta,\varphi) = \sqrt{\frac{\varepsilon(m)(2l+1)}{4\pi}\frac{(l-m)!}{(l+m)!}}\widetilde{P}_l^m(\cos(\vartheta))\begin{Bmatrix}\cos(m\varphi), \sigma = e \\ \sin(m\varphi), \sigma = o\end{Bmatrix} \tag{17.18}$$

They satisfy the following orthogonality conditions:

$$\int_0^{2\pi} d\varphi \int_{-\pi}^{+\pi} \sin(\vartheta)d\vartheta Y_{\sigma ml}(\vartheta,\varphi)Y_{\sigma' m' l'}(\vartheta,\varphi) = \delta_\sigma^{\sigma'}\delta_m^{m'}\delta_l^{l'} \tag{17.19}$$

$$\int_0^{2\pi} d\varphi \int_{-\pi}^{+\pi} \sin(\vartheta)d\vartheta \widetilde{Y}_{\sigma ml}(\vartheta,\varphi)\widetilde{Y}_{\sigma' m' l'}(\vartheta,\varphi) = (-1)^m\delta_\sigma^{\sigma'}\delta_m^{m'}\delta_l^{l'} \tag{17.20}$$

Using the following identity from Reference 9 evaluated at $\nu = 1/2$,

$$\exp[ix\cos(\alpha)] = \Gamma(\nu)(\nu/2)^{-\nu}\sum_{k=0}^{+\infty}(\nu+k)i^k J_{\nu+k}(x)C_k^{(\nu)}(\cos(\alpha)) \tag{17.21}$$

and the identities:

$$j_l(x) = \sqrt{\frac{\pi}{2x}}J_{k+1/2}(x) \tag{17.22}$$

$$C_k^{(1/2)}(x) = P_k(x) \tag{17.23}$$

one arrives at the following expansion of a plane wave in terms of regular spherical Bessel Functions:

$$\exp[+i\vec{k}\bullet\vec{r}] = \exp[ikr\cos(\gamma)] = \sum_{l=0}^{+\infty}(2l+1)i^l j_l(kr)P_l(\cos(\gamma)) \tag{17.24}$$

The angle γ is the angle between the two vectors $\vec{k}$ and $\vec{r}$. Let (ϑ,φ) and (α,β) be the polar coordinates of the direction of the position and wavenumber vectors, respectively. Then the angle γ is defined by the following relationship:

$$\cos(\gamma) = \cos(\alpha)\cos(\vartheta) + \sin(\alpha)\sin(\vartheta)\cos(\beta-\varphi) \tag{17.25}$$

Substituting the expansion of the Legendre polynomial given by Equation 17.11 into Equation 17.24, one arrives at the following expansion of a plane wave:

$$\exp[i\vec{k}\bullet\vec{r}] = \sum_{l=0}^{+\infty}(2l+1)i^k j_l(kr)\sum_{m=0}^{+l}\varepsilon(m)\frac{(l-m)!}{(l+m)!}P_l^m(\cos(\vartheta))P_l^m(\cos(\alpha))\cos(m(\beta-\varphi))$$
$$\sum_{l=0}^{+\infty}(2l+1)i^k j_l(kr)\sum_{m=0}^{+l}(-1)^m\varepsilon(m)\frac{(l-m)!}{(l+m)!}\widetilde{P}_l^m(\cos(\vartheta))\widetilde{P}_l^m(\cos(\alpha))\cos(m(\beta-\varphi)) \quad (17.26)$$

Expressing this equation in terms of the regular spherical wave functions and spherical harmonics, one arrives at the following expansion of the plane wave:

$$\exp[+i\vec{k}\bullet\vec{r}] =$$
$$4\pi\sum_{l=0}^{+\infty}i^l\sum_{m=0}^{+l}\sum_{\sigma=e}^{o}\mathrm{Re}\,\psi_{\sigma ml}(r)Y_{\sigma ml}(\alpha,\beta) = 4\pi\sum_{l=0}^{+\infty}i^l\sum_{m=0}^{+l}\sum_{\sigma=e}^{o}(-1)^m\,\mathrm{Re}\,\widetilde{\psi}_{\sigma ml}(r)\widetilde{Y}_{\sigma ml}(\alpha,\beta) \quad (17.27)$$

Using the orthogonality condition of the spherical harmonics and the expansion of a plane wave in terms of spherical wave functions given by Equation 17.27, one arrives at the following plane wave representation of the regular spherical wave functions:

$$\mathrm{Re}\,\psi_{\sigma ml}(r) = \frac{1}{4\pi i^l}\int_0^{2\pi}d\beta\int_{-\pi}^{+\pi}\sin(\alpha)d\alpha\exp[+i\vec{k}\bullet\vec{r}]Y_{\sigma ml}(\alpha,\beta) \quad (17.28)$$

$$\mathrm{Re}\,\widetilde{\psi}_{\sigma ml}(r) = \frac{1}{4\pi i^l}\int_0^{2\pi}d\beta\int_{-\pi}^{+\pi}\sin(\alpha)d\alpha\exp[+i\vec{k}\bullet\vec{r}]\widetilde{Y}_{\sigma ml}(\alpha,\beta) \quad (17.29)$$

The irregular (outgoing) spherical wave functions have the following integral representation:

$$\psi_{\sigma ml}(r) = \frac{1}{2\pi i^l}\int_0^{2\pi}d\beta\int_{C_\pm}\sin(\alpha)d\alpha\exp[+i\vec{k}\bullet\vec{r}]Y_{\sigma ml}(\alpha,\beta) \quad (17.30)$$

$$\widetilde{\psi}_{\sigma ml}(r) = \frac{1}{2\pi i^l}\int_0^{2\pi}d\beta\int_{C_\pm}\sin(\alpha)d\alpha\exp[+i\vec{k}\bullet\vec{r}]\widetilde{Y}_{\sigma ml}(\alpha,\beta) \quad (17.31)$$

The contour C_+ is used in the case of the inequality $z > 0$, and the contour C_- is used in the case $z < 0$.

The proof of Equation 17.30 follows. From the addition theorem of the spherical harmonics and the definition of the angle γ in Equation 17.25, one arrives at the following transformation:

$$P_l(\cos(\gamma)) = \frac{4\pi}{2l+1}\sum_{m=0}^{l}\sum_{\sigma=e}^{o} Y_{\sigma ml}(\vartheta,\varphi) Y_{\sigma ml}(\alpha,\beta) \tag{17.32}$$

Equation 17.32 rotates the vector (ϑ,φ) to $(0,0)$. Using the fact that the spherical harmonics form a linear representation of the rotation group, one may represent the effects of the rotation:

$$(\vartheta,\varphi) = R_z(\gamma')R_y(\beta')R_z(\alpha')(0,0) \tag{17.33}$$

on the spherical harmonics by the following unitary transformation:

$$Y_{\sigma ml} \circ R_{z''}(\gamma') \circ R_{y'}(\beta') \circ R_z(\alpha') = \sum_{m'=0}^{l}\sum_{\sigma'=e}^{o} D^{(l)}_{\sigma'm':\sigma m}(\alpha',\beta',\gamma')Y_{\sigma'm'l} \tag{17.34}$$

The rotation coefficients in Equation 17.34 are given by the following expression:

$$D^{(l)}_{\sigma'm':\sigma m}(\alpha',\beta',\gamma') = \delta^{e}_{\sigma'}\delta^{0}_{m'}\sqrt{\frac{4\pi}{2l+1}}Y_{\sigma ml}(\vartheta,\varphi) \tag{17.35}$$

From Equations 17.32 through 17.35 one arrives at the following result:

$$Y_{\sigma ml}(\alpha,\beta) = Y_{\sigma ml}(\vartheta,\varphi)P_l(\cos(\gamma)) \tag{17.36}$$

Substituting Equation 17.36 into Equation 17.30, one arrives at the following expression:

$$\begin{aligned}&\frac{1}{2\pi i^l}\int_0^{2\pi} d\beta \int_{C_\pm} \sin(\alpha)d\alpha \exp[+i\vec{k}\bullet\vec{r}]Y_{\sigma ml}(\alpha,\beta) = \\ &\frac{1}{2\pi i^l}Y_{\sigma ml}(\vartheta,\varphi)\int_0^{2\pi} d\beta \int_{C_\pm} \sin(a)d\alpha \exp[+ikr\cos(\gamma)]P_l(\cos(\gamma))\end{aligned} \tag{17.37}$$

Making a change of integration form the variables (α,β) to the variables (γ,β'') one obtains the result:

$$\begin{aligned}&\frac{1}{2\pi i^l}\int_0^{2\pi} d\beta \int_{C_\pm} \sin(\alpha)d\alpha \exp[+i\vec{k}\bullet\vec{r}]Y_{\sigma ml}(\alpha,\beta) = \\ &\frac{1}{2\pi i^l}Y_{\sigma ml}(\vartheta,\varphi)\int_0^{2\pi} d\beta'' \int_{C_\pm} \sin(\gamma)d\gamma \exp[+ikr\cos(\gamma)]P_l(\cos(\gamma))\end{aligned} \tag{17.38}$$

The above integrand is independent of the variable β'' and the integration over this variable is trivial:

$$\frac{1}{2\pi i^{l}}\int_{0}^{2\pi}d\beta\int_{C_\pm}\sin(\alpha)d\alpha\exp[+i\vec{k}\bullet\vec{r}]Y_{\sigma ml}(\alpha,\beta)=$$
$$\frac{2\pi}{2\pi i^{l}}Y_{\sigma ml}(\vartheta,\varphi)\int_{C_\pm}\sin(\gamma)d\gamma\exp[+ikr\cos(\gamma)]P_l(\cos(\gamma)) \tag{17.39}$$

Making the change of variables

$$Z=\cos(\gamma) \tag{17.40}$$

in Equation 17.39, one arrives at the following integral:

$$\frac{1}{2\pi i^{l}}\int_{0}^{2\pi}d\beta\int_{C_\pm}\sin(\alpha)d\alpha\exp[+i\vec{k}\bullet\vec{r}]Y_{\sigma ml}(\alpha,\beta)=$$
$$\frac{2\pi}{2\pi i^{l}}Y_{\sigma ml}(\vartheta,\varphi)\int_{i\infty}^{1}dZ\exp[+ikrZ]P_l(Z) \tag{17.41}$$

Substituting the Rodriguez Formula for the Legendre polynomials from Reference 9 into Equation 17.41:

$$P_l(Z)=\frac{1}{2^l l!}\frac{d^l}{dZ^l}(Z^2-1)^l \tag{17.42}$$

and integrating by parts one arrives at the following expression for the integral:

$$\frac{1}{2\pi i^{l}}\int_{0}^{2\pi}d\beta\int_{C_\pm}\sin(\alpha)d\alpha\exp[+i\vec{k}\bullet\vec{r}]Y_{\sigma ml}(\alpha,\beta)=$$
$$\frac{2\pi}{2\pi i^{l}}\frac{(ikr)^l}{2^l l!}Y_{\sigma ml}(\vartheta,\varphi)\int_{i\infty}^{1}dZ\exp[+ikrZ](1-Z^2)^l \tag{17.43}$$

Using the following integral representation of the spherical Hankel Function:

$$h_l^{(1)}(x)=\frac{x^l}{2^l l!}\int_{i\infty}^{1}dZ\exp[ixZ](1-Z^2)^l \tag{17.44}$$

one arrives at the result:

$$\psi_{\sigma ml}(r)=h_l^{(1)}(kr)Y_{\sigma ml}(\vartheta,\varphi)=\frac{1}{2\pi i^{l}}\int_{0}^{2\pi}d\beta\int_{C_\pm}\sin(\alpha)d\alpha\exp[+i\vec{k}\bullet\vec{r}]Y_{\sigma ml}(\alpha,\beta) \tag{17.45}$$

A similar derivation follows for Equation 17.31.

The free field Greens' Function has the following expansion in terms of the spherical wave functions:

$$G(\vec{r},\vec{r}':k) = \exp(ik\,|\,\vec{r}-\vec{r}'|)/4\pi\,|\,\vec{r}-\vec{r}'| = \frac{ik}{4\pi}h_0^{(1)}(k\,|\,\vec{r}-\vec{r}'|) = ik\sum_{\sigma ml}\psi_{\sigma ml}(r_>)\,\mathrm{Re}\,\psi_{\sigma ml}(r_<) \qquad (17.46)$$

This expansion follows from the following addition theorem for Bessel Functions from Reference 9 evaluated at $\nu = 1/2$:

$$H_\nu^{(1)}(w)/w^\nu = 2^\nu\Gamma(\nu)\sum_{k=0}^{\infty}(\nu+k)\frac{H_{\nu+k}^{(1)}(u)}{u^\nu}\frac{J_{\nu+k}(u')}{u'^\nu}C_k^{(\nu)}(\cos(\alpha)) \qquad (17.47)$$

$$w = \sqrt{u^2+u'^2-2uu'\cos(\alpha)} \qquad (17.48)$$

the following identities and the addition theorem for spherical harmonics given by Equation 17.32 are:

$$h_l^{(1)}(x) = \sqrt{\frac{\pi}{2x}}H_{l+1/2}^{(1)}(x) \qquad (17.49)$$

$$j_l(x) = \sqrt{\frac{\pi}{2x}}J_{l+1/2}(x) \qquad (17.50)$$

$$C_k^{(1/2)}(x) = P_k(x) \qquad (17.51)$$

18. TRANSFORMATION BETWEEN SPHERICAL AND CYLINDRICAL WAVE FUNCTIONS

This section discusses the transformations between spherical and cylindrical wave functions of the free field Helmholtz Equation.

The transformations from spherical to cylindrical wave functions are given by the following expressions:

$$\operatorname{Re}\chi_{\sigma m}^{(\pm)}(\alpha)=\sum_{l=m}^{\infty}B_{ml}^{(\pm)}(\alpha)\operatorname{Re}\psi_{\sigma ml} \tag{18.1}$$

$$B_{ml}^{(\pm)}(\alpha)=B_{ml}^{(\mp)}(\pi-\alpha)=(\pm i)^{l-m}\sqrt{\frac{2l+1}{2}\frac{(l-m)!}{(l+m)!}}P_l^m(\cos(\alpha)) \tag{18.2}$$

Equation 18.1 follows from the expansion of the free field Greens' Function in terms of spherical and cylindrical wave functions.

The transformations from cylindrical to spherical wave functions are given by the following expressions. The upper contour is used in Equations 18.4 and 18.5 if the z-coordinate is positive, and the lower contour is used if it is negative:

$$\operatorname{Re}\psi_{\sigma ml}=\int_0^{\pi}\sin(\alpha)d\alpha B_{ml}^{(\mp)}(\alpha)\operatorname{Re}\chi_{\sigma m}^{(\pm)}(\alpha) \tag{18.3}$$

$$\psi_{\sigma ml}=2\int_{C_\pm}\sin(\alpha)d\alpha B_{ml}^{(-)}(\alpha)\operatorname{Re}\chi_{\sigma m}^{(+)}(\alpha) \tag{18.4}$$

$$\psi_{\sigma ml}=2\int_{C_\mp}\sin(\alpha)d\alpha B_{ml}^{(+)}(\alpha)\operatorname{Re}\chi_{\sigma m}^{(-)}(\alpha) \tag{18.5}$$

$$\psi_{\sigma ml}=\int_{C}\sin(\alpha)d\alpha B_{ml}^{(\mp)}(\alpha)\chi_{\sigma m}^{(\pm)}(\alpha) \tag{18.6}$$

Equation 18.3 follows from Equations 18.1 and 18.2 and the orthogonality of the associated Legendre polynomials. The remaining transformations follow from the expansion of the Greens' Function. *Note: Equation 18.5 follows from Equation 18.4 under the transformation*:

$$\alpha\to\pi-\alpha \tag{18.7}$$

in the integrand. A discussion of these transformations may be found in References 10 through 12 by the author.

The proof of Equation 18.1 follows. Equations 16.12 and 17.27 give the expansion of a plane wave in terms of cylindrical and spherical wave functions. Equating these two expansions, one arrives at the following relationship between cylindrical and spherical wave functions:

$$\exp[+i\vec{k}\bullet\vec{r}] = 4\pi\sum_{l=0}^{+\infty} i^l \sum_{m=0}^{+l}\sum_{\sigma=e}^{o} \operatorname{Re}\psi_{\sigma ml}(r) Y_{\sigma ml}(\alpha,\beta) =$$
$$4\pi\sum_{m=0}^{+\infty}\sum_{\sigma}(\pm i)^m \operatorname{Re}\chi_{\sigma m}^{(\pm)}(r:\alpha)\sqrt{\frac{\varepsilon(m)}{2\pi}}\begin{Bmatrix}\cos(m\beta), \sigma = e\\ \sin(m\beta), \sigma = o\end{Bmatrix} \tag{18.8}$$

Using the orthogonality of the trigonometric basis set,

$$\sqrt{\frac{\varepsilon(m)}{2\pi}}\left\{\begin{matrix}\cos(m\beta), \sigma = e\\ \sin(m\beta), \sigma = o\end{matrix}\right. \tag{18.9}$$

Equation 18.1 is obtained upon multiplying both sides of Equation 18.8 by the basis given in Equation 18.9 and integrating with respect to β from 0 to 2π.

Equation 18.3 follows from Equation 18.1 and the orthonormality condition.

$$\int_0^{\pi}\sin(\alpha)d\alpha B_{ml}^{(\pm)}(\alpha)B_{ml'}^{(\mp)}(\alpha) = \delta_l^{l'} \tag{18.10}$$

Equations 18.4 and 18.5 follow from the expansions provided by Equations 16.56 and 17.46 of the free field Greens' Function in terms of cylindrical and spherical wave functions:

$$G(\vec{r},\vec{r}':k) = ik\sum_{\sigma ml}\psi_{\sigma ml}(r_>)\operatorname{Re}\psi_{\sigma ml}(r_<)$$
$$= 2ik\sum_{\sigma m}\int_{C_\pm}\operatorname{Re}\chi_{\sigma m}^{(+)}(r_>:\alpha)\operatorname{Re}\chi_{\sigma m}^{(-)}(r_<:\alpha)\sin(\alpha)d\alpha \tag{18.11}$$
$$= 2ik\sum_{\sigma m}\int_{C_\mp}\operatorname{Re}\chi_{\sigma m}^{(-)}(r_>:\alpha)\operatorname{Re}\chi_{\sigma m}^{(+)}(r_<:\alpha)\sin(\alpha)d\alpha$$

The upper contour is used in the case the quantity $(z_> - z_<) > 0$ is positive, and the lower contour is used if the quantity is negative. Substituting the expansion of the regular cylindrical wave function in terms spherical wave functions provided by Equation 18.1, one arrives at the following expression:

$$\begin{aligned} G(\vec{r},\vec{r}':k) &= ik\sum_{\sigma ml}\psi_{\sigma ml}(r_>)\operatorname{Re}\psi_{\sigma ml}(r_<) \\ &= 2ik\sum_{\sigma ml}\int_{C_\pm}\operatorname{Re}\chi^{(+)}_{\sigma m}(r_> : \alpha)B^{(-)}_{ml}(\alpha)\operatorname{Re}\psi_{\sigma ml}(r_<)\sin(\alpha)d\alpha \\ &= 2ik\sum_{\sigma ml}\int_{C_\mp}\operatorname{Re}\chi^{(-)}_{\sigma m}(r_> : \alpha)B^{(+)}_{ml}(\alpha)\operatorname{Re}\psi_{\sigma ml}(r_<)\sin(\alpha)d\alpha \end{aligned} \tag{18.12}$$

Equating the coefficients of the regular spherical wave function in Equation 18.12, one arrives at the following expressions for the representation of the outgoing spherical wave function in terms of regular cylindrical wave functions:

$$\begin{aligned} &\psi_{\sigma ml}(r) \\ &= 2\int_{C_\pm}\operatorname{Re}\chi^{(+)}_{\sigma m}(r : \alpha)B^{(-)}_{ml}(\alpha)\sin(\alpha)d\alpha \\ &= 2\int_{C_\mp}\operatorname{Re}\chi^{(-)}_{\sigma m}(r : \alpha)B^{(+)}_{ml}(\alpha)\sin(\alpha)d\alpha \end{aligned} \tag{18.13}$$

thus proving Equations 18.4 and 18.5.

Equation 18.6 follows from Equations 16.52 and 17.46, which provide the following expansion of the free field Greens' Function in terms of cylindrical and spherical wave functions:

$$G(\vec{r},\vec{r}':k) = ik\sum_{\sigma ml}\psi_{\sigma ml}(r_>)\operatorname{Re}\psi_{\sigma ml}(r_<) = ik\sum_{\sigma m}\int_{C}\chi^{(\pm)}_{\sigma m}(r_> : \alpha)\operatorname{Re}\chi^{(\mp)}_{\sigma m}(r_< : \alpha)\sin(\alpha)d\alpha \tag{18.14}$$

Substituting the expansion provided by Equation 18.1 of the regular cylindrical wave functions in terms of spherical wave functions one arrives at the following expansion of the free field Greens' Function:

$$G(\vec{r},\vec{r}':k) = ik\sum_{\sigma ml}\psi_{\sigma ml}(r_>)\operatorname{Re}\psi_{\sigma ml}(r_<) = ik\sum_{\sigma m}\int_{C}\chi^{(\pm)}_{\sigma m}(r_> : \alpha)B^{(\mp)}_{ml}(\alpha)\operatorname{Re}\psi_{\sigma ml}(r_<)\sin(\alpha)d\alpha \tag{18.15}$$

Equating the coefficients of the regular spherical wave functions, one arrives at the following expression:

$$\psi_{\sigma ml}(r) = \int_{C}\chi^{(\pm)}_{\sigma m}(r : \alpha)B^{(\mp)}_{ml}(\alpha)\sin(\alpha)d\alpha \tag{18.16}$$

thus proving Equation 18.6.

19. EXPANSION OF NORMAL MODE IN TERMS OF A SPHERICAL BASIS

This section describes the expansion of the normal mode representation of propagation in a waveguide in terms of a spherical basis set about the source and receiver points. Throughout this section, one will assume the density and sound speed are piece-wise constant as described in Sections 8 and 9.

From Equations 9.18 and 9.19 the normal mode representation of the Greens' Function in a homogeneously layered waveguide with a homogeneous half-space as a basement is given by the following normal mode and branch cut contributions:

$$G(r,z_s,z) = \frac{i}{4\rho(z_s)} \sum_n H_0^{(1)}(q_n r) F(z_s : q_n) F(z : q_n) + G_{Cut}(r,z_s,z) \tag{19.1}$$

$$G_{Cut}(r,z_s,z) = \frac{1}{8\pi} \int_{-\infty}^{+\infty} h_N dh_N H_0^{(1)}(qr) \{G_z(z,z_s : +h_N) - G_z(z,z_s : -h_N)\} \tag{19.2}$$

The following function is the depth function of the normal mode in the n'th layer:

$$F(z,q) = A_n^+ F_n^+(z:q) + A_n^- F_n^-(z:q), z_n \le z \le z_{n+1} \tag{19.3}$$

The functions:

$$F_n^{\pm}(z:q) = \exp(\pm i h_n (z - z_n)) \tag{19.4}$$

are the basis set of upward and downward-going waves in the n'th layer, and the coefficients $\{A_n^{\pm} : n = 0,1 \ldots N\}$ are the coefficients of the depth functions of a normal mode as described in Sections 8 and 9. The function $G_z(z,z_s)$ is the depth dependent Greens' Function defined in Section 8 and 9. In the case of a rigid basement the cut contribution vanishes.

Using the expansion:

$$H_\nu^{(1)}(w) = \sum_{k=-\infty}^{+\infty} H_{\nu+k}^{(1)}(u) J_k(u') \cos(k\varphi) \tag{19.5}$$

$$w = \sqrt{u^2 + u'^2 - 2uu'\cos(\varphi)} \tag{19.6}$$

from Reference 9, and the following properties for integral order Bessel Functions:

$$H_{-k}^{(1)}(x) = (-1)^k H_k^{(1)}(x) \tag{19.7}$$

$$J_{-k}(x) = (-1)^k J_k(x) \tag{19.8}$$

one obtains the following expansion of the normal mode contribution to the Greens' Function about the source point:

$$\begin{aligned}
&\frac{i}{4\rho(z_s)} H_0^{(1)}(q_n |\vec{r} - \Delta\vec{r}_s|) F(z:q_n) F(z_s + \Delta z_s) = \\
&\frac{i}{4\rho(z_s)} \sum_{m=0}^{\infty} \varepsilon(m) H_m^{(1)}(q_n r) J_m(q_n \Delta r_s) F(z:q_n) F(z_s + \Delta z_s) \cos(m\Delta\varphi_s)
\end{aligned} \tag{19.9}$$

Here, $(\Delta r_s, \Delta z_s, \Delta\varphi_s)$ are the polar coordinates of the displacement of the point about the source point $(0, z_s, 0)$, where the orientation of our coordinate system is such that the source and receiver points lie in the $\varphi = 0$ plane of the coordinate system. Expanding the depth functions in terms of upward and downward-going plane waves, one may express Equation 19.9 in terms of the product cylindrical wave functions about the source and receive points:

$$\begin{aligned}
&\frac{i}{4\rho(z_s)} H_0^{(1)}(q_n |\vec{r} - \Delta\vec{r}_s|) F(z:q_n) F(z_s + \Delta z_s) = \\
&\frac{2\pi i}{\rho(z_s)} \sum_{\sigma m} \sum_{s=+}^{-} \sum_{s'=+}^{-} G_{rcv}^{s}(z:q_n) G_{src}^{s'}(z_s:q_n) \chi_{\sigma m, N_{rcv}}^{(s)}(r, 0, 0:q_n) \operatorname{Re} \chi_{\sigma m, N_{src}}^{(s')}(\Delta r_s, \Delta z_s, \Delta\varphi_s : q_n)
\end{aligned} \tag{19.10}$$

The following functions are the coefficients of the expansion of the depth functions in terms of upward and downward-going plane waves about the source and receiver points, where N_{src} and N_{rcv} are the indices of the layers within which the source and receiver are located.

$$G_{src}^{\pm}(z_s:q_n) = A_{N_{src}}^{\pm} F_{N_{src}}^{\pm}(z_s:q_n) \tag{19.11}$$

$$G_{rcv}^{\pm}(z:q_n) = A_{N_{rcv}}^{\pm} F_{N_{rcv}}^{\pm}(z:q_n) \tag{19.12}$$

Define the following generalization of the expansion coefficients of the regular cylindrical wave functions in terms of regular spherical wave functions in the N'th layer:

$$B_{ml,N}^{\pm}(q) = (\pm i)^{l-m} \sqrt{\frac{2l+1}{2} \frac{(l-m)!}{(l+m)!}} (-q/k_N)^m \frac{d^m}{dx^m} P_l(x) \big|_{x = h_N(q)/k_N} \tag{19.13}$$

Equation 19.13 is equivalent to Equation 18.2 evaluated at the complex angle α_N defined by Equation 19.14, where k_N is the wave number in the N'th layer:

$$q = k_N \sin(\alpha_N) \tag{19.14}$$

In this case the vertical wave number is given by the following expression:

$$h_N(q) = +i\sqrt{q^2 - k_N^2} = k_N \cos(\alpha_N) \tag{19.15}$$

Substituting the following expansion of the regular cylindrical wave functions in terms of regular spherical wave functions into Equation 19.12:

$$\mathrm{Re}\,\chi^s_{\sigma m,N_{src}} = \sum_{l=m}^{\infty} B^s_{ml,N_{src}}(q)\,\mathrm{Re}\,\psi_{\sigma ml,N_{src}} \tag{19.16}$$

one arrives at the following expansion of the normal mode contribution to the Greens' Function in terms of a spherical basis set around the source point:

$$\begin{aligned} &\frac{i}{4\rho(z_s)} H_0^{(1)}(q_n|\vec{r} - \Delta\vec{r}_s|) F(z:q_n) F(z_s + \Delta z_s) = \\ &\sum_{\sigma ml} A_{\sigma ml,N_{src}}(r, z_s, z:q_n)\,\mathrm{Re}\,\psi_{\sigma ml,N_{src}}(\Delta r_s, \Delta z_s, \Delta\varphi_s) \end{aligned} \tag{19.17}$$

$$A_{\sigma ml,N_{src}}(r, z_s, z:q_n) = \frac{2\pi i}{\rho(z_s)} \sum_{s=+}^{-} \sum_{s'=+}^{-} G^s_{rcv}(z:q_n) G^{s'}_{src}(z_s:q_n) \chi^{(s)}_{\sigma m,N_{rcv}}(r,0,0:q_n) B^{s'}_{ml,N_{src}}(q_n) \tag{19.18}$$

Simplifying Equation 19.18, one arrives at the following expression:

$$\begin{aligned} &A_{\sigma ml,N_{src}}(r, z_s, z:q_n) = \\ &\delta^e_\sigma \frac{2\pi i}{\rho(z_s)} \sqrt{\frac{\varepsilon(m)}{8\pi}} H_m^{(1)}(q_n r) F_{N_{rcv}}(z:q_n) \{A^+_{N_{src}} F^+_{N_{src}}(z_s:q_n) B^+_{ml,N_{src}} + A^-_{N_{src}} F^-_{N_{src}}(z_s:q_n) B^-_{ml,N_{src}}\} \end{aligned} \tag{19.19}$$

The odd parity terms vanish in Equations 19.18 and 19.19 because one has chosen the orientation of our coordinate system such that the source and receiver points lie in the $\varphi = 0$ plane.

The expression for expanding the normal mode contribution about the receiver point instead of the source point is made by exchanging the source and receiver depths in Equations 19.18 and 19.19, except in the density term in these equations.

The expression for expanding the normal mode contribution about both the source and receiver points, where $(\Delta r_s, \Delta z_s, \Delta\varphi_s)$ is the displacement of the source point and $(\Delta r, \Delta z, \Delta\varphi)$ is the displacement of the receiver point follows from expanding Equation 19.19 about the receiver point.

$$A_{\sigma ml,N_{src}}(|\vec{r}+\Delta\vec{r}|,z_s,z+\Delta z:q_n)=$$
$$\delta_\sigma^e\frac{2\pi i}{\rho(z_s)}\sqrt{\frac{\varepsilon(m)}{8\pi}}H_m^{(1)}(q_n|\vec{r}+\Delta\vec{r}|)F_{N_{rcv}}(z+\Delta z:q_n) \tag{19.20}$$
$$\{A_{N_{src}}^+F_{N_{src}}^+(z_s:q_n)B_{ml,N_{src}}^+ + A_{N_{src}}^-F_{N_{src}}^-(z_s:q_n)B_{ml,N_{src}}^-\}$$

Using Equations 19.3, 19.4, and 19.5 to expand the Hankel Function and depth functions in Equation 19.20 about the receiver point, one arrives at the following expression:

$$A_{\sigma ml,N_{src}}(|\vec{r}+\Delta\vec{r}|,z_s,z+\Delta z:q_n)=$$
$$\delta_\sigma^e\frac{2\pi i}{\rho(z_s)}\sqrt{\frac{\varepsilon(m)}{8\pi}}\sum_{k=-\infty}^{+\infty}H_{m+k}^{(1)}(q_nr)J_k(q_n\Delta r)\cos(k(\pi-\Delta\varphi))$$
$$\{A_{N_{rcv}}^+F_{N_{rcv}}^+(z:q_n)\exp(+ih_{N_{rcv}}\Delta z)+A_{N_{rcv}}^-F_{N_{rcv}}^-(z:q_n)\exp(-ih_{N_{rcv}}\Delta z)\} \tag{19.21}$$
$$\{A_{N_{src}}^+F_{N_{src}}^+(z_s:q_n)B_{ml,N_{src}}^+ + A_{N_{src}}^-F_{N_{src}}^-(z_s:q_n)B_{ml,N_{src}}^-\}$$

The factor of π in the cosine function arises from the choice of orienting our coordinate system such that the direction of the line from the source point to the receiver point is in the direction $\varphi=0$. One can make the following rearrangement of the summation in Equation 19.21:

$$\sum_{k=-\infty}^{+\infty}H_{m+k}^{(1)}(qr)J_k(q\Delta r)\cos(k(\pi-\Delta\varphi))=$$
$$\sum_{m'=0}^{+\infty}\frac{\varepsilon(m')}{2}(H_{m+m'}^{(1)}(qr)+(-1)^{m'}H_{m-m'}^{(1)}(qr))J_{m'}(q\Delta r)\cos(m'(\pi-\Delta\varphi))= \tag{19.22}$$
$$\sum_{m'=0}^{+\infty}\frac{\varepsilon(m')}{2}(-1)^{m'}(H_{m+m'}^{(1)}(qr)+(-1)^{m'}H_{m-m'}^{(1)}(qr))J_{m'}(q\Delta r)\cos(m'\Delta\varphi)$$

Upon substitution of Equation 19.22 into Equation 19.21, one obtains the following expression for the expansion about the receiver point:

$$A_{\sigma ml,N_{src}}(|\vec{r}+\Delta\vec{r}|,z_s,z+\Delta z:q_n)=$$
$$\delta_\sigma^e\frac{2\pi i}{\rho(z_s)}\sqrt{\frac{\varepsilon(m)}{8\pi}}\sum_{m'=0}^{+\infty}(-1)^{m'}\frac{\varepsilon(m')}{2}(H_{m+m'}^{(1)}(q_nr)+(-1)^{m'}H_{m-m'}^{(1)}(q_nr))J_{m'}(q_n\Delta r)\cos(m'(\Delta\varphi))$$
$$\{A_{N_{rcv}}^+F_{N_{rcv}}^+(z:q_n)\exp(+ih_{N_{rcv}}\Delta z)+A_{N_{rcv}}^-F_{N_{rcv}}^-(z:q_n)\exp(-ih_{N_{rcv}}\Delta z)\} \tag{19.23}$$
$$\{A_{N_{src}}^+F_{N_{src}}^+(z_s:q_n)B_{ml,N_{src}}^+ + A_{N_{src}}^-F_{N_{src}}^-(z_s:q_n)B_{ml,N_{src}}^-\}$$

Making the following substitution into Equation 19.23:

$$\operatorname{Re}\chi_{\sigma'm',N_{rcv}}^{\pm}(\Delta r,\Delta z,\Delta\varphi)=\sqrt{\frac{\varepsilon(m')}{8\pi}}J_{m'}(q\Delta r)\exp(\pm ih\Delta z)\{\begin{matrix}\cos(m'\Delta\varphi),\sigma'=e\\ \sin(m'\Delta\varphi),\sigma'=o\end{matrix}\} \tag{19.24}$$

one arrives at an expansion in terms of the regular cylindrical wave functions about the receiver point:

$$A_{\sigma ml,N_{src}}(|\vec{r}+\Delta\vec{r}|,z_s,z+\Delta z:q_n)=$$
$$\sum_{s'=+}^{-}\sum_{\sigma'=e}^{o}\sum_{m'=0}^{+\infty}\delta_\sigma^e\delta_{\sigma'}^e\frac{2\pi i}{\rho(z_s)}\sqrt{\frac{\varepsilon(m)}{8\pi}}(-1)^{m'}4\pi\sqrt{\frac{\varepsilon(m')}{8\pi}}(H_{m+m'}^{(1)}(q_n r)+(-1)^{m'}H_{m-m'}^{(1)}(q_n r)) \tag{19.25}$$
$$\{A_{N_{src}}^{+}F_{N_{src}}^{+}(z_s:q_n)B_{ml,N_{src}}^{+}+A_{N_{src}}^{-}F_{N_{src}}^{-}(z_s:q_n)B_{ml,N_{src}}^{-}\}A_{N_{rcv}}^{s'}F_{N_{rcv}}^{s'}(z:q_n)\operatorname{Re}\chi_{\sigma'm',N_{rcv}}^{s'}(\Delta r,\Delta z,\Delta\varphi)$$

Substituting the expansion of the regular cylindrical wave function in terms of the regular spherical wave functions about the receiver point one obtains the following expansion:

$$A_{\sigma ml,N_{src}}(|\vec{r}+\Delta\vec{r}|,z_s,z+\Delta z:q_n)=\sum_{\sigma'm'l'}\Gamma_{\sigma ml,N_{src},\sigma'm'l',N_{rcv}}(r,z_s,z)\operatorname{Re}\psi_{\sigma'm'l',N_{rcv}}(\Delta r,\Delta z,\Delta\varphi) \tag{19.26}$$

$$\Gamma_{\sigma ml,N_{src},\sigma'm'l',N_{rcv}}(r,z_s,z)=\delta_\sigma^e\delta_{\sigma'}^e\frac{\pi i}{\rho(z_s)}(-1)^{m'}\sqrt{\varepsilon(m)\varepsilon(m')}(H_{m+m'}^{(1)}(q_n r)+(-1)^{m'}H_{m-m'}^{(1)}(q_n r))$$
$$\{A_{N_{src}}^{+}F_{N_{src}}^{+}(z_s:q_n)B_{ml,N_{src}}^{+}+A_{N_{src}}^{-}F_{N_{src}}^{-}(z_s:q_n)B_{ml,N_{src}}^{-}\} \tag{19.27}$$
$$\{A_{N_{rcv}}^{+}F_{N_{rcv}}^{+}(z:q_n)B_{ml,N_{rcv}}^{+}+A_{N_{rcv}}^{-}F_{N_{rcv}}^{-}(z:q_n)B_{ml,N_{rcv}}^{-}\}$$

The overall factor of $(-1)^{m'}$ in Equation 19.27 is due to the fact that the orientation of the coordinate systems at the source and receiver points are oriented such that the line from the source to the receiver point is in the direction $\varphi=0$, that is the orientation of the local coordinate systems at the source and receiver points are parallel.

Substituting Equation 19.26 into 19.17, one arrives at the following expression for the expansion of the normal mode component about both the source and receiver points:

$$\frac{i}{4\rho(z_s)}H_0^{(1)}(q_n|\vec{r}+\Delta\vec{r}-\Delta\vec{r}_s|)F(z+\Delta z:q_n)F(z_s+\Delta z_s)=$$
$$\sum_{\sigma ml}\sum_{\sigma'm'l'}\Gamma_{\sigma ml,N_{src},\sigma'm'l',N_{rcv}}(r,z_s,z:q_n)\operatorname{Re}\psi_{\sigma ml,N_{src}}(\Delta r_s,\Delta z_s,\Delta\varphi_s)\operatorname{Re}\psi_{\sigma ml,N_{rcv}}(\Delta r,\Delta z,\Delta\varphi) \tag{19.28}$$

Here, $(\Delta r_s,\Delta z_s,\Delta\varphi_s)$ is the displacement of the source point, and $(\Delta r,\Delta z,\Delta\varphi)$ is the displacement of the receiver point.

A similar expansion may be made for the integrand of the branch cut contribution, where the coefficients $\{A_N^{\pm}\}$ of the depth functions are replaced by the expansion coefficients $\{\bar{A}_N^{\pm},\bar{A}_{N'}^{\pm}\}$ of the depth dependent Green's Function in the integrand.

The generalization of Equations 19.19 and 19.27 to local coordinate systems whose z-axis are vertical but whose orientation of the x-axis is not parallel to the line from the source to the receiver are given by the following expressions:

$$A_{\sigma ml,N_{src}}(r,z_s,z,\varphi_s:q_n)=\frac{2\pi i}{\rho(z_s)}\sqrt{\frac{\varepsilon(m)}{8\pi}}H_m^{(1)}(q_nr)F_{N_{rcv}}(z:q_n)\left\{\begin{matrix}\cos(m\varphi_s),\sigma=e\\ \sin(m\varphi_s),\sigma=o\end{matrix}\right\}$$
$$\{A^+_{N_{src}}F^+_{N_{src}}(z_s:q_n)B^+_{ml,N_{src}}+A^-_{N_{src}}F^-_{N_{src}}(z_s:q_n)B^-_{ml,N_{src}}\} \tag{19.29}$$

$$\Gamma_{\sigma ml,N_{src},\sigma'm'l',N_{rcv}}(r,z_s,z,\varphi_s,\varphi)=\frac{\pi i}{\rho(z_s)}\sqrt{\varepsilon(m)\varepsilon(m')}(H^{(1)}_{m+m'}(q_nr)+(-1)^{m'}H^{(1)}_{m-m'}(q_nr))$$
$$\{A^+_{N_{src}}F^+_{N_{src}}(z_s:q_n)B^+_{ml,N_{src}}+A^-_{N_{src}}F^-_{N_{src}}(z_s:q_n)B^-_{ml,N_{src}}\}\left\{\begin{matrix}\cos(m\varphi_s),\sigma=e\\ \sin(m\varphi_s),\sigma=o\end{matrix}\right\} \tag{19.30}$$
$$\{A^+_{N_{rcv}}F^+_{N_{rcv}}(z:q_n)B^+_{ml,N_{rcv}}+A^-_{N_{rcv}}F^-_{N_{rcv}}(z:q_n)B^-_{ml,N_{rcv}}\}\left\{\begin{matrix}\cos(m'\varphi),\sigma'=e\\ \sin(m'\varphi),\sigma'=o\end{matrix}\right\}$$

Here φ_s is the angle between the x-axis of the local coordinate system at the source and the location of the receiver as seen from the source, and φ is the angle between the x-axis of the local coordinate system at the receiver and the location of the source as seen from the receiver. Note, in Equations 19.19 and 19.27 these two angles have the following values: $\varphi_s=0$, and $\varphi=\pi$.

20. METHOD FOR INCLUDING DIRECTIVITY OF THE SOURCE AND THE RECEIVER IN THE NORMAL MODE TERM

This section uses the results of Section 19 to include the effects of the directivity of the source and the receiver in the normal mode description of propagation.

Let $D_{Src}(\vartheta,\varphi)$ denote the beam pattern for the source, and $D_{Rcv}(\vartheta,\varphi)$ denote the beam pattern of the receiver. Using the orthogonality of the spherical harmonics, one can project the beam patterns for the source and receiver onto the spherical harmonics by performing the following surface integrals:

$$d_{Src,\sigma ml} = \int_0^{2\pi} d\varphi \int_0^{\pi} \sin(\vartheta) d\vartheta Y_{\sigma ml}(\vartheta,\varphi) D_{Src}(\vartheta,\varphi) \tag{20.1}$$

$$d_{Rcv,\sigma ml} = \int_0^{2\pi} d\varphi \int_0^{\pi} \sin(\vartheta) d\vartheta Y_{\sigma ml}(\vartheta,\varphi) D_{Rcv}(\vartheta,\varphi) \tag{20.2}$$

These integrals can easily be performed for arbitrary beam patterns by using Gauss Legendre integration of sufficiently high order. Using the expansion of the normal mode contribution about the source and receiver points given by Equation 19.28:

$$\begin{aligned}&\frac{i}{4\rho(z_s)} H_0^{(1)}(q_n |\vec{r} + \Delta\vec{r} - \Delta\vec{r}_s|) F(z+\Delta z : q_n) F(z_s + \Delta z_s) = \\ &\sum_{\sigma ml} \sum_{\sigma' m' l'} \Gamma_{\sigma ml, N_{src}, \sigma' m' l', N_{rcv}}(r, z_s, z, \varphi_s, \varphi : q_n) \operatorname{Re} \psi_{\sigma ml, N_{src}}(\Delta r_s, \Delta z_s, \Delta\varphi_s) \operatorname{Re} \psi_{\sigma ml, N_{rcv}}(\Delta r, \Delta z, \Delta\varphi)\end{aligned} \tag{20.3}$$

one may project the above normal mode contribution onto the directivity pattern of the source and receiver by performing the following summations:

$$\sum_{\sigma ml} \sum_{\sigma' m' l'} \Gamma_{\sigma ml, N_{src}, \sigma' m' l', N_{rcv}}(r, z_s, z, \varphi_s, \varphi : q) d_{Src,\sigma ml} d_{Rcv,\sigma' m' l'} \tag{20.4}$$

The coefficients $\{\Gamma_{\sigma ml, N_{src}, \sigma' m' l', N_{rcv}}(r, z_s, z, \varphi_s, \varphi : q)\}$ are defined in Equation 19.30.

In the case the receiver is omni directional, the projection of the normal mode onto the directivity of the source is given by the following summation:

$$\sum_{\sigma ml} A_{\sigma ml, N_{src}}(r, z_s, z, \varphi_s : q) d_{Src,\sigma ml} \tag{20.5}$$

The coefficients $\{A_{\sigma ml,N_{src}}(r,z_s,z,\varphi_s:q)\}$ are the expansion coefficients of the normal mode contribution about the source point defined in Equation 19.29.

Equations 20.4 and 20.5 assume the coordinate system used in computing the projection of the source and receiver beam pattern onto a spherical basis are parallel to the coordinate system used in computing the coefficients in Equations 19.29, and 19.30. In the event these coordinate systems are not parallel, the computation of projection of the source and receiver beam patterns onto a spherical basis set will require rotating the origin of the argument (ϑ,φ) of the directivity functions.

21. ROTATION OF COORDINATE SYSTEMS

This section describes the procedure for rotating the local coordinate systems at the source and the receiver for an arbitrary orientation. A practical introduction to the theory of representations of the rotation group "SO(3)" is provided by Tinkham[13]. More advanced treatises on representation theory are provided in texts by Helgason[14] and Weyl[15].

The conventions of Rose are used to describe an arbitrary rotation as a series of three rotations about the fixed axes (x,y,z):

$$R(\alpha,\beta,\gamma)=R_z(\alpha)\circ R_y(\beta)\circ R_z(\gamma) \tag{21.1}$$

The equivalent rotation about relative axes is given by the following set of three rotations about the rotated axes:

$$R(\alpha,\beta,\gamma)=R_z(\alpha)\circ R_y(\beta)\circ R_z(\gamma)=R_{z''}(\gamma)\circ R_{y'}(\beta)\circ R_z(\alpha) \tag{21.2}$$

Here, (x',y',z') are the axes obtained by rotating the axes (x,y,z) by the rotation $R_z(\gamma)$ about the z-axis, (x'',y'',z'') are the axes obtained by further rotating the axes by the rotation $R_{y'}(\beta)$ about the y'-axis, and (x''',y''',z''') are the axes obtained by further rotating these axes by the rotation $R_{z''}(\alpha)$ about the z''-axis.

The complex spherical harmonics are defined as follows:

$$Y_{ml}(\vartheta,\varphi)=\sqrt{\frac{2l+1}{4\pi}\frac{(l-m)!}{(l+m)!}}\exp(+im\varphi)P_l^m(\cos(\vartheta)), m=-l,-(l-1)...+l \tag{21.3}$$

Assume the Condon-Shortley phase convention for negative indices given below:

$$P_l^{-m}(\cos(\vartheta))=(-1)^m P_l^m(\cos(\vartheta)) \tag{21.4}$$

$$Y_{-ml}(\vartheta,\varphi)=(-1)^m Y_{ml}^*(\vartheta,\varphi) \tag{21.5}$$

The spherical harmonics $\{Y_{ml}(\vartheta,\varphi): m=-l,-(l-1)...+l\}$ form a $(2l+1)$-dimensional irreducible representation of the rotation group "SO(3)" on the homogeneous manifold SO(3)/SO(2)~S^2. The spherical harmonics obey the following transformation rules under an arbitrary rotation:

$$R(\alpha,\beta,\gamma)*(Y_{ml})(\vartheta,\varphi)=Y_{ml}(R(\alpha,\beta,\gamma)(\vartheta,\varphi))=\sum_{m'=-l}^{+l}Y_{m'l}(\vartheta,\varphi)D_{m',m}^{(l)}(\alpha,\beta,\gamma) \tag{21.6}$$

The Wigner rotation coefficients $D^{(j)}_{m,m'}(\alpha,\beta,\gamma)$ are given by the following relationships from Reference 13:

$$D^{(j)}_{m',m}(\alpha,\beta,\gamma) = \exp(-im'\alpha)\exp(-im\gamma)d^{(j)}_{m',m}(\beta) \tag{21.7}$$

$$d^{(j)}_{m',m}(\beta) =$$
$$\sum_k (-1)^k \frac{\sqrt{(j+m)!(j-m)!(j+m')!(j-m')!}}{k!(j+m-k)!(j-m'-k)!(k+m'-m)!}(\cos(\beta/2))^{2j-2k-m'+m}(-\sin(\beta/2))^{2k+m'-m} \tag{21.8}$$

The summation in Equation 21.8 extends overall possible values of the index k for which the denominator is finite, that is, k satisfies the following inequalities:

$$k \geq 0 \tag{21.9a}$$

$$j+m-k \geq 0 \tag{21.9b}$$

$$j-m'-k \geq 0 \tag{21.9c}$$

$$k+m'-m \geq 0 \tag{21.9d}$$

Equation 21.9 results in the following limits for the index k:

$$\min(j+m, j-m') \geq k \geq \max(m-m',0) \tag{21.10}$$

The Wigner coefficients form an irreducible, unitary representation of the group Spin(3)~SU(2) corresponding to the covering group of the rotation group "SO(3)". "SU(2)" is the Z_2-valued covering group of the rotation group "SO(3)" associated with the spinor representation of the rotation group. One of the properties of the group "SU(2)" is that the rotation of a spin 1/2 representation by 2π results in a multiplication of the spinor by -1.

The relationship between group elements in "SU(2)" and "SO(3)" is given by the following correspondence:

$$u(\alpha,\beta,\gamma) \in SU(2) \rightarrow u(\alpha,\beta,\gamma)\sigma_i u^+(\alpha,\beta,\gamma) = R_{i,j}(\alpha,\beta,\gamma)\sigma_j \tag{21.11}$$

Here, $u(\alpha,\beta,\gamma) \in SU(2)$ is the following group element:

$$u(\alpha,\beta,\gamma) = \begin{pmatrix} +\exp(+i(\alpha+\gamma)/2)\cos(\beta/2) & +\exp(+i(\gamma-\alpha)/2)\sin(\beta/2) \\ -\exp(-i(\gamma-\alpha)/2)\sin(\beta/2) & +\exp(-i(\alpha+\gamma)/2)\cos(\beta/2) \end{pmatrix} \tag{21.12}$$

The Hermitian matrices $\{\sigma_i : i = 1,2,3\}$ are the following Pauli spin matrices:

$$\sigma_1 = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \tag{21.13a}$$

$$\sigma_2 = \begin{pmatrix} 0 & -i \\ +i & 0 \end{pmatrix} \tag{21.13b}$$

$$\sigma_3 = \begin{pmatrix} +1 & 0 \\ 0 & -1 \end{pmatrix} \tag{21.13c}$$

The matrix $R_{i,j}(\alpha,\beta,\gamma)$ is the three-dimensional rotation matrix with the following elements:

$$R_{1,1}(\alpha,\beta,\gamma) = +\cos(\alpha)\cos(\beta)\cos(\gamma) - \sin(\alpha)\sin(\gamma) \tag{21.14a}$$

$$R_{1,2}(\alpha,\beta,\gamma) = +\sin(\alpha)\cos(\beta)\cos(\gamma) + \cos(\alpha)\sin(\gamma) \tag{21.14b}$$

$$R_{1,3}(\alpha,\beta,\gamma) = -\sin(\beta)\cos(\gamma) \tag{21.14c}$$

$$R_{2,1}(\alpha,\beta,\gamma) = -\cos(\alpha)\cos(\beta)\sin(\gamma) - \sin(\alpha)\cos(\gamma) \tag{21.14d}$$

$$R_{2,2}(\alpha,\beta,\gamma) = -\sin(\alpha)\cos(\beta)\sin(\gamma) + \cos(\alpha)\cos(\gamma) \tag{21.14e}$$

$$R_{2,3}(\alpha,\beta,\gamma) = +\sin(\beta)\sin(\gamma) \tag{21.14f}$$

$$R_{3,1}(\alpha,\beta,\gamma) = +\cos(\alpha)\sin(\beta) \tag{21.14g}$$

$$R_{3,2}(\alpha,\beta,\gamma) = +\sin(\alpha)\sin(\beta) \tag{21.14h}$$

$$R_{3,3}(\alpha,\beta,\gamma) = +\cos(\beta) \tag{21.14i}$$

This matrix is given by the following rotations about the z-axis and y-axis:

$$R_z(\gamma)R_y(\beta)R_z(\alpha) =$$
$$\begin{pmatrix} +\cos(\gamma) & +\sin(\gamma) & 0 \\ -\sin(\gamma) & +\cos(\gamma) & 0 \\ 0 & 0 & 1 \end{pmatrix}\begin{pmatrix} +\cos(\beta) & 0 & -\sin(\beta) \\ 0 & 1 & 0 \\ +\sin(\beta) & 0 & +\cos(\beta) \end{pmatrix}\begin{pmatrix} +\cos(\alpha) & +\sin(\alpha) & 0 \\ -\sin(\alpha) & +\cos(\alpha) & 0 \\ 0 & 0 & 1 \end{pmatrix} \tag{21.15}$$

The correspondence between the Wigner coefficients and the matrix representation of the corresponding group element is given by the following relationship:

$$D^{(j)}_{m,m'}(\alpha,\beta,\gamma) = U^{(j)}_{m,m'}(R_z(-\gamma)R_y(-\beta)R_z(-\alpha)) = U^{(j)}_{m,m'}((R_z(\alpha)R_y(\beta)R_z(\gamma))^{-1}) \tag{21.16}$$

Here, $\{U^{(l)}_{m,m'}(g): m,m' = -l,\ldots+l\}$ are the matrix elements of a $(2l+1)$-dimensional unitary representation of the group element $g \in SO(3) \subset SU(2)$. Strictly speaking, this matrix is a representation of the covering group "SU(2)". The group "SU(2)" is a non-trivial Z_2-valued principal fiber bundle over the group "SO(3)". However, in the case of integral angular momentum, the above matrix elements are constant along the fiber of this principal bundle, and the matrix elements may be considered to be functions of the base space "SO(3)". The constant nature of this matrix along the fiber follows from the fact that for integral angular momentum a rotation by multiples of 2π leaves the representation unchanged.

The action of the group element $g \in SO(3)$ acting on the spherical harmonics can be expressed in the following form in terms of this unitary representation:

$$L_g{}^*(Y_{ml}) = \sum_{m'} Y_{m'l} U^{(l)}_{m',m}(g^{-1}) \tag{21.17}$$

The function:

$$L_g : SO(3)/SO(2) \to SO(3)/SO(2) \tag{21.18}$$

$$L_g[g'] \equiv [g \bullet g'] = \{g \bullet g' \bullet h : h \in SO(2)\} \tag{21.19}$$

denotes the natural left action

$$SO(3) x SO(3) \to SO(3) \to SO(3)/SO(2) \approx S^2 \tag{21.20}$$

$$(g,g') \to g \bullet g' \to g \bullet [g'] \equiv [g \bullet g'] \tag{21.21}$$

of the group element $g \in SO(3)$ acting on the homogeneous space

$$\pi : SO(3) \to SO(3)/SO(2) \approx S^2 \tag{21.22}$$

$$g \to \pi(g) \equiv [g] = \{g \bullet h : h \in SO(2)\} \tag{21.23}$$

and $L_g{}^*$ denotes the pullback of this function.

One may introduce the following transformation from the complex spherical harmonics: $\{Y_{ml}(\vartheta,\varphi): m = -l,\ldots+l\}$ to the real spherical harmonics $\{Y_{\sigma ml}(\vartheta,\varphi): \sigma = e,o; m = 0,\ldots l\}$:

$$Y_{eml}(\vartheta,\varphi)=\frac{\sqrt{\varepsilon(m)}}{2}(Y_{ml}(\vartheta,\varphi)+(-1)^m Y_{-ml}(\vartheta,\varphi)) \tag{21.24}$$

$$Y_{oml}(\vartheta,\varphi)=\frac{\sqrt{\varepsilon(m)}}{2i}(Y_{ml}(\vartheta,\varphi)-(-1)^m Y_{-ml}(\vartheta,\varphi)) \tag{21.25}$$

The effect of the rotation $R(\alpha,\beta,\gamma)$ on the real spherical harmonics is of the form:

$$R(\alpha,\beta,\gamma)^*(Y_{\sigma ml})(\vartheta,\varphi)=\sum_{\sigma'=e}^{o}\sum_{m'=0}^{l} Y_{\sigma' m' l}(\vartheta,\varphi) D^{(l)}_{\sigma' m';\sigma m}(\alpha,\beta,\gamma) \tag{21.26}$$

The matrix elements of the real Wigner rotation coefficients are given by the following relationships:

$$D^{(l)}_{em',em}=\frac{1}{2}\sqrt{\frac{\varepsilon(m)}{\varepsilon(m')}}\{D^{(l)}_{m',m}+(-1)^{m'}D^{(l)}_{-m',m}+(-1)^{m}D^{(l)}_{m',-m}+(-1)^{m+m'}D^{(l)}_{-m',-m}\} \tag{21.27}$$

$$D^{(l)}_{om',em}=\frac{+i}{2}\sqrt{\frac{\varepsilon(m)}{\varepsilon(m')}}\{D^{(l)}_{m',m}-(-1)^{m'}D^{(l)}_{-m',m}+(-1)^{m}D^{(l)}_{m',-m}-(-1)^{m+m'}D^{(l)}_{-m',-m}\} \tag{21.28}$$

$$D^{(l)}_{em',om}=\frac{-i}{2}\sqrt{\frac{\varepsilon(m)}{\varepsilon(m')}}\{D^{(l)}_{m',m}+(-1)^{m'}D^{(l)}_{-m',m}-(-1)^{m}D^{(l)}_{m',-m}-(-1)^{m+m'}D^{(l)}_{-m',-m}\} \tag{21.29}$$

$$D^{(l)}_{om',om}=\frac{1}{2}\sqrt{\frac{\varepsilon(m)}{\varepsilon(m')}}\{D^{(l)}_{m',m}-(-1)^{m'}D^{(l)}_{-m',m}-(-1)^{m}D^{(l)}_{m',-m}+(-1)^{m+m'}D^{(l)}_{-m',-m}\} \tag{21.30}$$

From Equations 21.7 and 21.8 one may derive the following relationships by rearranging the terms in the summation in Equation 21.8:

$$D^{(j)}_{m',m}(\alpha,\beta,\gamma)=\exp(-im'\alpha)\exp(-im\gamma)d^{(j)}_{m',m}(\beta) \tag{21.31}$$

$$d^{(j)}_{-m',-m}(\beta)=(-1)^{m+m'}d^{(j)}_{+m',+m}(\beta) \tag{21.32}$$

$$d^{(j)}_{-m',+m}(\beta)=(-1)^{m+m'}d^{(j)}_{+m',-m}(\beta) \tag{21.33}$$

Substituting Equations 21.31 through 21.33 into Equations 21.27 through 21.30, one arrives at the following representation of the real Wigner rotation coefficients:

$$D^{(j)}_{em',em}=\sqrt{\frac{\varepsilon(m)}{\varepsilon(m')}}\{d^{(j)}_{m',m}(\beta)\cos(m'\alpha+m\gamma)+(-1)^{m'}d^{(j)}_{-m',m}(\beta)\cos(m'\alpha-m\gamma)\} \tag{21.34}$$

$$D^{(j)}_{om',em} = \sqrt{\frac{\varepsilon(m)}{\varepsilon(m')}}\{d^{(j)}_{m',m}(\beta)\sin(m'\alpha + m\gamma) + (-1)^{m'} d^{(j)}_{-m',m}(\beta)\sin(m'\alpha - m\gamma)\} \tag{21.35}$$

$$D^{(j)}_{em',om} = -\sqrt{\frac{\varepsilon(m)}{\varepsilon(m')}}\{d^{(j)}_{m',m}(\beta)\sin(m'\alpha + m\gamma) - (-1)^{m'} d^{(j)}_{-m',m}(\beta)\sin(m'\alpha - m\gamma)\} \tag{21.36}$$

$$D^{(j)}_{om',om} = \sqrt{\frac{\varepsilon(m)}{\varepsilon(m')}}\{d^{(j)}_{m',m}(\beta)\cos(m'\alpha + m\gamma) - (-1)^{m'} d^{(j)}_{-m',m}(\beta)\cos(m'\alpha - m\gamma)\} \tag{21.37}$$

Consider the case j = 1. The spherical harmonics are given by the following relationships:

$$Y_{11} = -\sqrt{\frac{3}{8\pi}}\sin(\vartheta)e^{+i\varphi} = -\sqrt{\frac{3}{8\pi}}\frac{x+iy}{r} \tag{21.38}$$

$$Y_{01} = +\sqrt{\frac{3}{4\pi}}\cos(\vartheta) = +\sqrt{\frac{3}{4\pi}}\frac{z}{r} \tag{21.39}$$

$$Y_{-11} = +\sqrt{\frac{3}{8\pi}}\sin(\vartheta)e^{-i\varphi} = +\sqrt{\frac{3}{8\pi}}\frac{x-iy}{r} \tag{21.40}$$

In the case of the rotation $R(\alpha,0,0) = R_z(\alpha)$ about the z-axis, the Wigner rotation coefficients in the case j=1 are given by the following relationships:

$$D^{(1)}_{m',m}(\alpha,0,0) = \begin{pmatrix} \exp(-i\alpha) & 0 & 0 \\ 0 & +1 & 0 \\ 0 & 0 & \exp(+i\alpha) \end{pmatrix} \tag{21.41}$$

In this case, the rotated spherical harmonics are given by the following linear combination of the spherical harmonics:

$$Y_{11} \circ R_z(\alpha) = e^{-i\alpha}Y_{11} \tag{21.42a}$$

$$Y_{01} \circ R_z(\alpha) = Y_{01} \tag{21.42b}$$

$$Y_{-11} \circ R_z(\alpha) = e^{+i\alpha}Y_{-11} \tag{21.42c}$$

In the case of the rotation $R(0,\beta,0) = R_y(\beta)$ the Wigner rotation coefficients in the case j=1 are given by the following relationships:

$$D^{(1)}_{m',m}(0,\beta,0) =$$

$$\begin{pmatrix} \cos(\beta/2)^2 & -\sqrt{2}\cos(\beta/2)\sin(\beta/2) & \sin(\beta/2)^2 \\ +\sqrt{2}\cos(\beta/2)\sin(\beta/2) & \cos(\beta/2)^2-\sin(\beta/2)^2 & -\sqrt{2}\cos(\beta/2)\sin(\beta/2) \\ \sin(\beta/2)^2 & +\sqrt{2}\cos(\beta/2)\sin(\beta/2) & \cos(\beta/2)^2 \end{pmatrix} = \tag{21.43}$$

$$\begin{pmatrix} (1+\cos(\beta))/2 & -\sqrt{2}\sin(\beta)/2 & (1-\cos(\beta))/2 \\ +\sqrt{2}\sin(\beta)/2 & \cos(\beta) & -\sqrt{2}\sin(\beta)/2 \\ (1-\cos(\beta))/2 & +\sqrt{2}\sin(\beta)/2 & (1+\cos(\beta))/2 \end{pmatrix}$$

The rotated spherical harmonics are given by the following linear combination of the spherical harmonics:

$$Y_{11} \circ R_y(\beta) = -\sqrt{\frac{3}{8\pi}}\frac{(x\cos(\beta)-z\sin(\beta)+iy)}{r} =$$
$$\frac{(1+\cos(\beta))}{2}Y_{11} + \frac{\sqrt{2}}{2}\sin(\beta)Y_{01} + \frac{(1-\cos(\beta))}{2}Y_{-11} \tag{21.44a}$$

$$Y_{01} \circ R_y(\beta) = +\sqrt{\frac{3}{4\pi}}\frac{(x\sin(\beta)+z\cos(\beta))}{r} =$$
$$-\frac{\sqrt{2}}{2}\sin(\beta)Y_{11} + \cos(\beta)Y_{01} + \frac{\sqrt{2}}{2}\sin(\beta)Y_{-11} \tag{21.44b}$$

$$Y_{-11} \circ R_y(\beta) = +\sqrt{\frac{3}{8\pi}}\frac{(x\cos(\beta)-z\sin(\beta)-iy)}{r} =$$
$$\frac{(1-\cos(\beta))}{2}Y_{11} - \frac{\sqrt{2}}{2}\sin(\beta)Y_{01} + \frac{(1+\cos(\beta))}{2}Y_{-11} \tag{21.44c}$$

22. HELMHOLTZ INTEGRAL EQUATION

This section reviews the Helmholtz Integral Equation in anticipation of the next section on the T-matrix description of scattering from a target.

The free-field Greens' Function for the Helmholtz differential equation is given by the following expression:

$$G(\vec{r},\vec{r}':k) = \frac{\exp(+ik\,|\vec{r}-\vec{r}'|)}{4\pi\,|\vec{r}-\vec{r}'|} = ik\sum_{\sigma ml}\psi_{\sigma ml}(\vec{r}_>)\,\mathrm{Re}\,\psi_{\sigma ml}(\vec{r}_<) \qquad (22.1)$$

This function satisfies the following inhomogeneous partial differential equation:

$$(\vec{\nabla}\bullet\vec{\nabla}+k^2)G(\vec{r},\vec{r}':k) = -\delta^3(\vec{r}-\vec{r}') \qquad (22.2)$$

Let $S_\infty \cup S_0$ denote the bounding surface of the volume V, where the surface S_∞ is the surface at spatial infinity, and S_0 is the union of the surfaces of all scatterers in the volume. Let the function

$$\psi = \psi_{Incident} + \psi_{Scattered} \qquad (22.3)$$

denote the solution of the Helmholtz Equation in the volume V resulting from the scattering of the incident field $\psi_{Incident}$ from the scatterers interior to this volume. The incident field is given by the following surface integral over the sphere at spatial infinity:

$$\psi_{Incident}(\vec{r}) = \iint_{S_\infty} d\vec{A}\bullet\{G(\vec{r},\vec{r}':k)\vec{\nabla}\psi(\vec{r}') - \psi(\vec{r}')\vec{\nabla}G(\vec{r},\vec{r}':k)\} \qquad (22.4)$$

Using Greens' Theorem, one obtains the following integral equation for the total field:

$$\begin{aligned}
&\int_{S_\infty} d\vec{A} \bullet \{G(\vec{r},\vec{r}':k)\vec{\nabla}\psi(\vec{r}') - \vec{\nabla}G(\vec{r},\vec{r}':k)\psi(\vec{r}')\} \\
&- \int_{S_0} d\vec{A} \bullet \{G(\vec{r},\vec{r}':k)\vec{\nabla}\psi(\vec{r}') - \vec{\nabla}G(\vec{r},\vec{r}':k)\psi(\vec{r}')\} \\
&= \int_V dV \{G(\vec{r},\vec{r}':k)\vec{\nabla}\bullet\vec{\nabla}\psi(\vec{r}') - \vec{\nabla}\bullet\vec{\nabla}G(\vec{r},\vec{r}':k)\psi(\vec{r}')\} \qquad (22.5) \\
&= -\int_V dV \{\vec{\nabla}\bullet\vec{\nabla}G(\vec{r},\vec{r}':k) + k^2 G(\vec{r},\vec{r}':k)\}\psi(\vec{r}') \\
&= \begin{cases} \psi(\vec{r}), \vec{r} \in V \\ \frac{1}{2}\psi(\vec{r}), \vec{r} \in \partial V \\ 0, \vec{r} \notin V \cup \partial V \end{cases}
\end{aligned}$$

Substituting Equation 22.4 for the surface integral over the sphere at spatial infinity, one obtains the following integral equation for the field in terms of the incident field and the surface integral over the surface of the scatterers:

$$\psi_{Incident}(\vec{r}) - \int_{S_0} d\vec{A} \bullet \{G(\vec{r},\vec{r}':k)\vec{\nabla}\psi(\vec{r}') - \vec{\nabla}G(\vec{r},\vec{r}':k)\psi(\vec{r}')\} = \begin{cases} \psi(\vec{r}), \vec{r} \in V \\ \frac{1}{2}\psi(\vec{r}), \vec{r} \in \partial V \\ 0, \vec{r} \notin V \cup \partial V \end{cases} \qquad (22.6)$$

Equation 22.6 is the Helmholtz Integral Equation for the acoustic field.

23. SCATTERING FROM A RIGID TARGET

This section gives a basic description of the spherical T-matrix approach to scattering from a rigid target. One of the first articles to appear on the use of the spherical T-matrix description of acoustic scattering is the article by Waterman[16]. Hackman[17] provides a review of the spherical and spheroidal T-matrix methods.

In the following derivation, one may make use of the following bilinear expansion of the free-field Greens' Function in terms of spherical wave functions:

$$G(\vec{r},\vec{r}':k) = ik\sum_{\sigma ml}\psi_{\sigma ml}(\vec{r}_>)\,\mathrm{Re}\,\psi_{\sigma ml}(\vec{r}_<) \tag{23.1}$$

Let the function

$$\psi = \psi_{Incident} + \psi_{Scattered} \tag{23.2}$$

denote a solution of the Helmholtz Equation representing the acoustic field in the presence of a rigid target. The acoustic field is required to satisfy the following boundary condition on the surface of the rigid target:

$$\hat{n}\bullet\vec{\nabla}\psi|_S = 0 \tag{23.3}$$

The incident and scattered fields satisfy the following Helmholtz Integral Equations:

$$\psi_{Incident}(\vec{r}) = \iint_{S_\infty} d\vec{A}\bullet\{G(\vec{r},\vec{r}':k)\vec{\nabla}\psi(\vec{r}') - \psi(\vec{r}')\vec{\nabla}G(\vec{r},\vec{r}':k)\} \tag{23.4}$$

$$\psi_{Incident}(\vec{r}) - \int_{S_0} d\vec{A}\bullet\{G(\vec{r},\vec{r}':k)\vec{\nabla}\psi(\vec{r}') - \vec{\nabla}G(\vec{r},\vec{r}':k)\psi(\vec{r}')\} = \begin{cases}\psi(\vec{r}), \vec{r}\in V\\ \frac{1}{2}\psi(\vec{r}), \vec{r}\in\partial V\\ 0, \vec{r}\notin V\cup\partial V\end{cases} \tag{23.5}$$

Substituting Equation 23.3 for the normal gradient of the acoustic field on the surface of the target into Equation 23.5, one arrives at the following integral equation:

$$\psi_{Incident}(\vec{r}) + \int_{S_0} d\vec{A}\bullet\vec{\nabla}G(\vec{r},\vec{r}':k)\psi(\vec{r}') = \begin{cases}\psi(\vec{r}), \vec{r}\in V\\ \frac{1}{2}\psi(\vec{r}), \vec{r}\in\partial V\\ 0, \vec{r}\notin V\cup\partial V\end{cases} \tag{23.6}$$

The incident field and scattered field may be expanded in terms of the regular and outgoing spherical wave functions:

$$\psi_{Incident} = \sum_{\sigma ml} a_{\sigma ml} \operatorname{Re}\psi_{\sigma ml} \tag{23.7}$$

$$\psi_{Scattered} = \sum_{\sigma ml} b_{\sigma ml} \psi_{\sigma ml} \tag{23.8}$$

The scattered field on the surface of the target may be expanded in terms of the regular spherical wave functions:

$$\psi_{Scattered} = \sum_{\sigma ml} \beta_{\sigma ml} \psi_{\sigma ml} \tag{23.9}$$

Substituting Equations 23.7 and 23.9 into Equation 23.2, one arrives at the following expansion of the acoustic field on the surface of the target:

$$\psi|_S = \sum_{\sigma ml} (a_{\sigma ml} + \beta_{\sigma ml}) \operatorname{Re}\psi_{\sigma ml}|_S \tag{23.10}$$

Substituting Equations 23.1 and 23.10 into Equation 23.6 evaluated at a point interior to the target, one arrives at the following linear equation:

$$\sum_{\sigma ml} a_{\sigma ml} \operatorname{Re}\psi_{\sigma ml} + \sum_{\sigma ml} \operatorname{Re}\psi_{\sigma ml} \sum_{\sigma'm'l'} Q_{\sigma ml,\sigma'm'l'} (a_{\sigma'm'l'} + \beta_{\sigma'm'l'}) = 0 \tag{23.11}$$

The matrix Q is the infinite dimensional matrix defined by the following surface integral over the surface of the target:

$$Q_{\sigma ml,\sigma'm'l'} = -\iint_S d\vec{A} \bullet \vec{\nabla} \psi_{\sigma ml} \operatorname{Re}\psi_{\sigma'm'l'} \tag{23.12}$$

Substituting Equations 23.1 and 23.10 into Equation 23.6 evaluated at a point exterior to the target, one arrives at the following linear equation:

$$\sum_{\sigma ml} b_{\sigma ml} \psi_{\sigma ml} - \sum_{\sigma ml} \psi_{\sigma ml} \sum_{\sigma'm'l'} \operatorname{Re}Q_{\sigma ml,\sigma'm'l'} (a_{\sigma'm'l'} + \beta_{\sigma'm'l'}) = 0 \tag{23.13}$$

The matrix ReQ is the infinite dimensional matrix defined by the following surface integral over the surface of the target:

$$\operatorname{Re}Q_{\sigma ml,\sigma'm'l'} = -\iint_S d\vec{A} \bullet \vec{\nabla} \operatorname{Re}\psi_{\sigma ml} \operatorname{Re}\psi_{\sigma'm'l'} \tag{23.14}$$

The solution of this pair of infinite dimensional linear equations is given by the following expression:

$$b_{\sigma ml} = \sum_{\sigma' m' l'} T_{\sigma ml, \sigma' m' l'} a_{\sigma' m' l'} \tag{23.15}$$

Here, the matrix T is the T-matrix of the scattering from the target defined by the following matrix product

$$T_{\sigma ml, \sigma' m' l'} = -\sum_{\sigma'' m'' l''} \operatorname{Re} Q_{\sigma ml, \sigma'' m'' l''} Q^{-1}{}_{\sigma'' m'' l'', \sigma' m' l'} \tag{23.16}$$

In the case of a rigid sphere, the T-matrix is diagonal and has the following form in terms of the derivative of the regular spherical Bessel Function and Hankel Function:

$$T_{\sigma ml, \sigma' m' l'} = -\delta_{\sigma ml}^{\sigma' m' l'} \frac{j_l'(ka)}{h_l^{(1)}{}'(ka)} \tag{23.17}$$

24. SCATTERING FROM A SOLID ELASTIC TARGET

This section describes the scattering from a solid, elastic target using the spherical T-matrix methods described by Hackman[17].

In order to describe the scattering from an elastic target it is necessary to introduce the vector Helmholtz Equation for an elastic solid. Let us begin by describing the Helmholtz Equation for a fluid in terms of Newton's' Second Law of Motion and the stress tensor for a homogeneous, isotropic fluid:

$$\rho \frac{d^2\vec{u}}{dt^2} = \vec{\nabla} \bullet \vec{T} \tag{24.1}$$

$$T_{ij} = -g_{ij} P \tag{24.2}$$

Equation 24.1 is Newton's' Second Law of Motion for a fluid in terms of the displacement (u_i) of the fluid and the stress tensor of the fluid (T_{ij}). Equation 24.2 is the stress tensor of a homogeneous isotropic fluid, where P is the pressure of the fluid, and g_{ij} is the metric tensor of the coordinate system. Substituting Equation 24.2 into Equation 24.1 and assuming a harmonic dependence on time, one arrives at the following expression for the displacement of the fluid in terms of the gradient of the pressure:

$$\vec{u} = \frac{1}{\rho\omega^2} \vec{\nabla} P \tag{24.3}$$

Analogous equations of motion for a homogeneous, isotropic, elastic solid are given below:

$$\rho \frac{d^2\vec{u}}{dt^2} = \vec{\nabla} \bullet \vec{T} \tag{24.4}$$

$$T_{ij} = g_{ij} \lambda \vec{\nabla} \bullet \vec{u} + \mu(\nabla_i u_j + \nabla_j u_i) \tag{24.5}$$

Here, the parameters λ and μ are the compression and shear components of the Lame' constants of the material. The longitudinal and shear velocities are given by the following expressions in terms of the Lame' constants and the density of the material:

$$v_L = \sqrt{\frac{\lambda + 2\mu}{\rho}} \tag{24.6}$$

$$v_T = \sqrt{\frac{\mu}{\rho}} \tag{24.7}$$

Substituting Equation 24.5 into Equation 24.4, one arrives at the following equation of motion for the elastic solid:

$$\begin{aligned}\rho\frac{d^2u_i}{dt^2} &= g^{jk}\nabla_k T_{ji} = (\lambda+2\mu)\nabla_i(\vec{\nabla}\bullet\vec{u}) + \mu(\vec{\nabla}\bullet\vec{\nabla}(u_i) - \nabla_i(\vec{\nabla}\bullet u)) \\ &= (\lambda+2\mu)\nabla_i(\vec{\nabla}\bullet\vec{u}) - \mu(\vec{\nabla}\times(\vec{\nabla}\times u))_i\end{aligned} \tag{24.8}$$

In the case of an elastic solid, it is often convenient to express the displacement vector in terms of the Debye potentials Φ and $\vec{A}$:

$$\vec{u} = \vec{\nabla}\Phi + \vec{\nabla}\times\vec{A} \tag{24.9}$$

Substituting Equation 24.9 into Equation 24.8, one arrives at the following equation of motion for the Debye potentials:

$$\rho\frac{d^2}{dt^2}(\vec{\nabla}\Phi + \vec{\nabla}\times\vec{A}) = (\lambda+2\mu)\vec{\nabla}(\vec{\nabla}\bullet\vec{\nabla}\Phi) - \mu(\vec{\nabla}\times(\vec{\nabla}\times(\vec{\nabla}\times\vec{A}))) \tag{24.10}$$

This equation of motion is equivalent to the following pair of differential equations, since the longitudinal and transverse components of the vector in Equation 24.10 are uncoupled:

$$\rho\frac{d^2}{dt^2}(\vec{\nabla}\Phi) = (\lambda+2\mu)\vec{\nabla}(\vec{\nabla}\bullet\vec{\nabla}\Phi) \tag{24.11}$$

$$\rho\frac{d^2}{dt^2}(\vec{\nabla}\times\vec{A}) = -\mu(\vec{\nabla}\times(\vec{\nabla}\times(\vec{\nabla}\times\vec{A}))) \tag{24.12}$$

Extracting an overall gradient from Equation 24.11, and curl from Equation 24.12, one may express this pair of equations in the following form:

$$\rho\frac{d^2}{dt^2}\Phi = (\lambda+2\mu)\vec{\nabla}\bullet\vec{\nabla}\Phi \tag{24.13}$$

$$\rho\frac{d^2}{dt^2}\vec{A} = -\mu(\vec{\nabla}\times(\vec{\nabla}\times\vec{A})) \tag{24.14}$$

Let $\vec{u}$ and $\vec{w}$ be a pair of solutions of the equations of motions for an elastic solid occupying the volume V. The following surface integral over the boundary of this volume is equal to zero:

$$\iiint_V dV\{u_i \nabla_j T^{ij}(w) - w_i \nabla_j T^{ij}(u)\} = \iiint_V dV \nabla_j \{u_i T^{ij}(w) - w_i T^{ij}(u)\} =$$
$$\iint_{\partial V} dA n_j \{u_i T^{ij}(w) - w_i T^{ij}(u)\} = \iint_{\partial V} dA\{\vec{u} \bullet \vec{t}(w) - \vec{w} \bullet \vec{t}(u)\} = 0 \tag{24.15}$$

Here, the vector

$$t_i(u) = T_{ij}(u) n^j \tag{24.16}$$

is the surface traction at the boundary of the volume V, where the vector $\hat{n}$ is the outgoing normal to the surface ∂V. Equation 24.15 is the third Betti Identity for the equations of motion for the elastic solid. This identity is used extensively in the construction of the T-matrix description of an elastic target.

The boundary conditions between a pair of elastic solids are that the displacements and surface tractions be continuous at the boundary.

$$\vec{u}_+ = \vec{u}_- \tag{24.17}$$

$$\vec{t}(u_+) = \vec{t}(u_-) \tag{24.18}$$

Here, $\vec{u}_+$ and $\vec{u}_-$ are the displacement vectors on the two sides of the boundary evaluated at the boundary. In the limit the shear modulus of both solids approach zero, the above boundary conditions become the following boundary conditions between two fluids, respectively.

$$\frac{1}{\rho_+} \vec{n} \bullet \vec{\nabla} P_+ = \frac{1}{\rho_-} \vec{n} \bullet \vec{\nabla} P_- \tag{24.19}$$

$$P_+ = P_- \tag{24.20}$$

The boundary conditions between a fluid and a solid are that the normal components of the displacement and surface traction are continuous, and the tangential surface traction of the solid vanishes at the boundary.

$$\vec{u}_+ \bullet \vec{n} = \vec{u}_- \bullet \vec{n} \tag{24.21}$$

$$\vec{t}(u_+) \bullet \vec{n} = \vec{t}(u_-) \bullet \vec{n} \tag{24.22}$$

$$\vec{t}(u_+) \times \vec{n} = \vec{t}(u_-) \times \vec{n} = 0 \tag{24.23}$$

The tangential surface traction of an inviscid fluid is zero. Equations 24.3 and 24.2 give the displacement vector and stress tensor of an inviscid fluid in terms of pressure.

The vector spherical wave functions:

$$\vec{V}_{1,\sigma ml} = \vec{\nabla} \times (\vec{r}\, \psi^{T}_{\sigma ml}(r)) \tag{24.24a}$$

$$\vec{V}_{2,\sigma ml} = \frac{1}{k_T} \vec{\nabla} \times \vec{V}_{1,\sigma ml} \tag{24.24b}$$

$$\vec{V}_{3,\sigma ml} = \frac{1}{k_L} \vec{\nabla} \psi^{L}_{\sigma ml} \tag{24.24c}$$

are the analogues of the scalar spherical wave functions. The functions

$$\psi^{L}_{\sigma ml}(r) = h^{(1)}_{l}(k_L r) Y_{\sigma ml}(\vartheta, \varphi) \tag{24.25a}$$

$$\psi^{T}_{\sigma ml}(r) = h^{(1)}_{l}(k_T r) Y_{\sigma ml}(\vartheta, \varphi) \tag{24.25b}$$

are the scalar spherical harmonics for the longitudinal and transverse components of the acoustic field. They are solutions of the following scalar Helmholtz Equations:

$$\vec{\nabla} \bullet \vec{\nabla} \psi^{L}_{\sigma ml} + k^{2}_{L} \psi^{L}_{\sigma ml} = 0 \tag{24.26a}$$

$$\vec{\nabla} \bullet \vec{\nabla} \psi^{T}_{\sigma ml} + k^{2}_{T} \psi^{T}_{\sigma ml} = 0 \tag{24.26b}$$

The first two vector spherical wave functions are solutions of the transverse wave equation, and the third vector spherical wave function is a solution of the longitudinal wave equation.

The incident and scattered fields may be expanded in terms of the regular and outgoing spherical wave functions, respectively:

$$\psi_{Incident} = \sum_{\sigma ml} a_{\sigma ml} \operatorname{Re} \psi_{\sigma ml} \tag{24.27}$$

$$\psi_{Scattered} = \sum_{\sigma ml} b_{\sigma ml} \psi_{\sigma ml} \tag{24.28}$$

The displacements and surface tractions interior to the target may be expanded in terms of the regular vector spherical wave functions:

$$\vec{u}_{-} = \sum_{\tau=1}^{3} \sum_{\sigma ml} \mathrm{A}_{\tau,\sigma ml} \operatorname{Re} V_{\tau,\sigma ml} \tag{24.29}$$

$$\vec{t}_{-} = \sum_{\tau=1}^{3} \sum_{\sigma ml} \mathrm{A}_{\tau,\sigma ml}\, t(\operatorname{Re} V_{\tau,\sigma ml}) \tag{24.30}$$

The displacements and surface tractions on the fluid side of the interface may be expressed in the following form:

$$\vec{u}_+ = \frac{1}{\rho_+ \omega^2} \vec{\nabla}(\psi_{Incident} + \psi_{Scattered}) \tag{24.31}$$

$$\vec{t}_+ = (\psi_{Incident} + \psi_{Scattered})\hat{n} \tag{24.32}$$

The displacements and surface tractions on the fluid and elastic solid sides of the boundary are related by the following boundary conditions:

$$\vec{u}_+ \bullet \hat{n} = \vec{u}_- \bullet \hat{n} \tag{24.33}$$

$$\vec{t}_+ \bullet \hat{n} = \vec{t}_- \bullet \hat{n} \tag{24.34}$$

$$\vec{t}_- \times \hat{n} = 0 \tag{24.35}$$

Substituting Equations 24.31 and 24.32 into Equations 24.33 and 24.34, one obtains the following expressions for the acoustic field and its normal gradient at the boundary in terms of the elastic field in the interior of the target:

$$\psi|_S = \vec{t}_+ \bullet \hat{n} = \vec{t}_- \bullet \hat{n} \tag{24.36}$$

$$\hat{n} \bullet \vec{\nabla}\psi|_S = \rho_+ \omega^2 \vec{u}_+ \bullet \hat{n} = \rho_+ \omega^2 \vec{u}_- \bullet \hat{n} \tag{24.37}$$

The scattered field satisfies the following integral equation:

$$\int_S d\vec{A} \bullet \{G(\vec{r},\vec{r}':k)\vec{\nabla}\psi(\vec{r}') - \vec{\nabla}G(\vec{r},\vec{r}':k)\psi(\vec{r}')\} = \begin{cases} +\psi_{Scattered}(\vec{r}), \vec{r} \notin V \\ -\psi_{Incident}(\vec{r}), \vec{r} \in V \end{cases} \tag{24.38}$$

Substituting Equations 24.36 and 24.37 into Equation 24.38, one obtains the following integral equation for the scattered field:

$$\int_S dA\{G(\vec{r},\vec{r}':k)\rho_+\omega^2 u_- \bullet \hat{n} - \hat{n} \bullet \vec{\nabla}G(\vec{r},\vec{r}':k)\vec{t}_- \bullet \hat{n}\} = \begin{cases} +\psi_{Scattered}(\vec{r}), \vec{r} \notin V \\ -\psi_{Incident}(\vec{r}), \vec{r} \in V \end{cases} \tag{24.39}$$

Substituting the bilinear expansion of the free field Greens' Function

$$G(\vec{r},\vec{r}':k) = ik \sum_{\sigma ml} \psi_{\sigma ml}(\vec{r}_>) \operatorname{Re}\psi_{\sigma ml}(\vec{r}_<) \tag{24.40}$$

and the expansions for the normal displacements and surface traction given by Equations 24.29 and 24.30 into Equation 24.39 evaluated at a point exterior to the scatterer, one obtains the following linear equation for the scattered field:

$$b_{\sigma ml} = \sum_{\tau'=1}^{3} \sum_{\sigma' m' l'} \mathrm{Re}\, Q_{\sigma ml, \tau'\sigma' m' l'} A_{\tau', \sigma' m' l'} \tag{24.41}$$

$$\mathrm{Re}\, Q_{\sigma ml, \tau'\sigma' m' l'} = ik \iint_S dA \{\rho_+ \omega^2 \,\mathrm{Re}\, \psi_{\sigma ml} \hat{n} \bullet \mathrm{Re}\, \vec{V}_{\tau', \sigma' m' l'} - \hat{n} \bullet \vec{\nabla}\, \mathrm{Re}\, \psi_{\sigma ml} \hat{n} \bullet \vec{t}\,(\mathrm{Re}\, V_{\tau', \sigma' m' l'})\} \tag{24.42}$$

Evaluating Equation 24.39 at a point interior to the scatterer, one obtains the following linear equation for the incident field:

$$a_{\sigma ml} = \sum_{\tau'=1}^{3} \sum_{\sigma' m' l'} Q_{\sigma ml, \tau'\sigma' m' l'} A_{\tau', \sigma' m' l'} \tag{24.43}$$

$$Q_{\sigma ml, \tau'\sigma' m' l'} = ik \iint_S dA \{\rho_+ \omega^2 \psi_{\sigma ml} \hat{n} \bullet \mathrm{Re}\, \vec{V}_{\tau', \sigma' m' l'} - \hat{n} \bullet \vec{\nabla}\, \psi_{\sigma ml} \hat{n} \bullet \vec{t}\,(\mathrm{Re}\, V_{\tau', \sigma' m' l'})\} \tag{24.44}$$

The interior fields satisfy Betti's Third Identity:

$$\iint_S dA \{\mathrm{Re}\, \vec{V}_{\tau, \sigma ml} \bullet \vec{t}_- - u_- \bullet \vec{t}\,(\mathrm{Re}\, \vec{V}_{\tau, \sigma ml})\} = 0 \tag{24.45}$$

Using the identity

$$\vec{u}_- = \hat{n} \times (\hat{n} \times \vec{u}_-) - (\vec{u}_- \bullet \hat{n})\hat{n} = \hat{n} \times (\hat{n} \times \vec{u}_-) - (\vec{u}_+ \bullet \hat{n})\hat{n} \tag{24.46}$$

one can express Equation 23.45 in the following form:

$$\iint_S dA \{\mathrm{Re}\, \vec{V}_{\tau, \sigma ml} \bullet \vec{t}_- - (\hat{n} \bullet \vec{u}_+)(\hat{n} \bullet \vec{t}\,(\mathrm{Re}\, \vec{V}_{\tau, \sigma ml})) + (\hat{n} \times (\hat{n} \times u_-)) \bullet \vec{t}\,(\mathrm{Re}\, \vec{V}_{\tau, \sigma ml})\} = 0 \tag{24.47}$$

Assume the acoustic field on the surface of the target has the following expansion in terms of the regular spherical wave functions:

$$\psi_+ = \sum_{\sigma ml} \beta_{\sigma ml}\, \mathrm{Re}\, \psi_{\sigma ml} \tag{24.48}$$

In this case, the displacements on the fluid side of the interface have the following expansion:

$$\vec{u}_+ = \frac{1}{\rho_+ \omega^2} \sum_{\sigma ml} \beta_{\sigma ml} \vec{\nabla}\, \mathrm{Re}\, \psi_{\sigma ml} \tag{24.49}$$

Substituting the above expansion into Equation 24.47, one arrives at the following linear equation between the expansion coefficients of the fields on the fluid and target sides of the interface:

$$\sum_{\sigma'm'l'} P_{\tau\sigma ml,\sigma'm'l'}\beta_{\sigma'm'l'} - \sum_{\tau'=1}^{3}\sum_{\sigma'm'l'} R_{\tau\sigma ml,\tau'\sigma'm'l'}A_{\tau'\sigma'm'l'} = 0 \tag{24.50}$$

$$P_{\tau\sigma ml,\sigma'm'l'} = \iint_S dA\{(\hat{n}\bullet\vec{t}(\mathrm{Re}\vec{V}_{\tau\sigma ml}))(\hat{n}\bullet\vec{\nabla}\,\mathrm{Re}\psi_{\sigma'm'l'})\} \tag{24.51}$$

$$R_{\tau\sigma ml,\tau'\sigma'm'l'} = \iint_S dA\{\mathrm{Re}\vec{V}_{\tau\sigma ml}\bullet\vec{t}(\mathrm{Re}\vec{V}_{\tau'\sigma'm'l'}) - \vec{t}(\mathrm{Re}\vec{V}_{\tau\sigma ml})\bullet(\hat{n}\times(\hat{n}\times\mathrm{Re}\vec{V}_{\tau'\sigma'm'l'}))\} \tag{24.52}$$

Combining Equations 24.41, 24.43, and 24.50, one obtains the following representation for the T-matrix:

$$T = \mathrm{Re}QR^{-1}P(QR^{-1}P)^{-1} \tag{24.53}$$

The expansion coefficients for the interior fields are given by the following linear equation in terms of the expansion coefficients of the incident field:

$$A = R^{-1}P(QR^{-1}P)^{-1}a \tag{24.54}$$

An alternative formulation of the T-matrix for a solid target is obtained by using the boundary conditions to express Equations 24.39 and 24.45 in the following form:

$$\int_S dA\{G(\vec{r},\vec{r}':k)\rho_+\omega^2 u_-\bullet\hat{n} - \hat{n}\bullet\vec{\nabla}G(\vec{r},\vec{r}':k)\vec{t}_+\bullet\hat{n}\} = \begin{cases} +\psi_{Scattered}(\vec{r}), \vec{r}\notin V \\ -\psi_{Incident}(\vec{r}), \vec{r}\in V \end{cases} \tag{24.55}$$

$$\iint_S dA\{\mathrm{Re}\vec{V}_{\tau,\sigma ml}\bullet\vec{t}_+ - u_-\bullet\vec{t}(\mathrm{Re}\vec{V}_{\tau,\sigma ml})\} = 0 \tag{24.56}$$

Suppose the surface traction at the surface of the target has the following expansion in terms of the spherical harmonics:

$$\hat{n}\bullet t_+ = \sum_{\sigma ml}\beta_{\sigma ml}Y_{\sigma ml} \tag{24.57}$$

Evaluating Equation 24.55 at a point exterior to the target, one obtains the following linear equation for the scattered field:

$$b_{\sigma ml} = \sum_{\tau=1}^{3}\sum_{\sigma'm'l'}\mathrm{Re}\,Q_{\sigma ml,\tau\sigma'm'l'}A_{\tau,\sigma'm'l'} - \mathrm{Re}\,M_{\sigma ml,\sigma'm'l'}\beta_{\sigma'm'l'} \tag{24.58}$$

$$\mathrm{Re}\,Q_{\sigma ml,\tau'\sigma'm'l'} = ik\rho_+\omega^2\iint_S dA\,\mathrm{Re}\psi_{\sigma ml}\,\mathrm{Re}\vec{V}_{\tau',\sigma'm'l'} \tag{24.59}$$

$$\operatorname{Re} M_{\sigma ml,\sigma' m' l'} = ik \iint_S dA \hat{n} \bullet \vec{\nabla} \operatorname{Re} \psi_{\sigma ml} Y_{\sigma' m' l'} \tag{24.60}$$

Evaluating Equation 24.55 at a point interior to the target one obtains the following linear equation for the incident field:

$$a_{\sigma ml} = -\sum_{\tau=1}^{3} \sum_{\sigma' m' l'} Q_{\sigma ml, \tau\sigma' m' l'} A_{\tau,\sigma' m' l'} + M_{\sigma ml,\sigma' m' l'} \beta_{\sigma' m' l'} \tag{24.61}$$

$$Q_{\sigma ml,\tau'\sigma' m' l'} = ik\rho_{+}\omega^{2} \iint_S dA\, \psi_{\sigma ml} \operatorname{Re} \vec{V}_{\tau',\sigma' m' l'} \tag{24.62}$$

$$M_{\sigma ml,\sigma' m' l'} = ik \iint_S dA \hat{n} \bullet \vec{\nabla} \psi_{\sigma ml} Y_{\sigma' m' l'} \tag{24.63}$$

Evaluating Equation 24.56, one obtains the following linear equation between the interior and exterior fields:

$$\sum_{\sigma' m' l'} P_{\tau\sigma ml,\sigma' m' l'} \beta_{\sigma' m' l'} - \sum_{\tau'=1}^{3} \sum_{\sigma' m' l'} R_{\tau\sigma ml,\tau'\sigma' m' l'} A_{\tau',\sigma' m' l'} = 0 \tag{24.64}$$

$$P_{\tau\sigma ml,\sigma' m' l'} = \iint_S dA (\hat{n} \bullet \operatorname{Re} \vec{V}_{\tau,\sigma ml}) Y_{\sigma' m' l'} \tag{24.65}$$

$$R_{\tau\sigma ml,\tau'\sigma' m' l'} = \iint_S dA \vec{t} (\operatorname{Re} \vec{V}_{\tau,\sigma ml}) \bullet \operatorname{Re} \vec{V}_{\tau',\sigma' m' l'} \tag{24.66}$$

Combining Equations 24.58, 24.61, and 24.64 one obtains the following expression for the T-matrix:

$$T = -(\operatorname{Re} M - \operatorname{Re} Q R^{-1} P)(M - Q R^{-1} P)^{-1} \tag{24.67}$$

25. SCATTERING FROM AN ELASTIC TARGET IN A WAVEGUIDE

This section uses the results of Sections 19 through 24 to describe the scattering from an elastic target in a waveguide. This section assumes the scattering from the target is described by the spherical T-matrix of the target.

The free-field scattering from the target is of the following form:

$$\psi_{Scattered} = \sum_{\sigma ml} \sum_{\sigma' m' l'} \psi_{\sigma ml} T_{\sigma ml, \sigma' m' l'} a_{\sigma' m' l'} \tag{25.1}$$

The coefficients $a_{\sigma ml}$ are the expansion coefficients of the incident wave in terms of regular spherical wave functions

$$\psi_{Incident} = \sum_{\sigma ml} a_{\sigma ml} \operatorname{Re} \psi_{\sigma ml} \tag{25.2}$$

In Section 19, the following expansion of the normal mode contribution to the waveguide Greens' Function was derived in the following form:

$$\begin{gathered} \frac{i}{4\rho(z_s)} H_0^{(1)}(q_n |\vec{r} + \Delta\vec{r} - \Delta\vec{r}_s|) F(z+\Delta z : q_n) F(z_s + \Delta z_s) = \\ \sum_{\sigma ml} \sum_{\sigma' m' l'} \Gamma_{\sigma ml, N_{src}, \sigma' m' l', N_{rcv}}(r, z_s, z : q_n) \operatorname{Re} \psi_{\sigma ml, N_{src}}(\Delta r_s, \Delta z_s, \Delta \varphi_s) \operatorname{Re} \psi_{\sigma ml, N_{rcv}}(\Delta r, \Delta z, \Delta \varphi) \end{gathered} \tag{25.3}$$

The coefficients of this expansion are defined as follows:

$$\begin{gathered} \Gamma_{\sigma ml, N_{src}, \sigma' m' l', N_{rcv}}(r, z_s, z, \varphi_s, \varphi) = \frac{\pi i}{\rho(z_s)} \sqrt{\varepsilon(m)\varepsilon(m')} (H_{m+m'}^{(1)}(q_n r) + (-1)^{m'} H_{m-m'}^{(1)}(q_n r)) \\ \{A_{N_{src}}^{+} F_{N_{src}}^{+}(z_s : q_n) B_{ml, N_{src}}^{+} + A_{N_{src}}^{-} F_{N_{src}}^{-}(z_s : q_n) B_{ml, N_{src}}^{-}\} \{ {\cos(m\varphi_s), \sigma = e \atop \sin(m\varphi_s), \sigma = o} \} \\ \{A_{N_{rcv}}^{+} F_{N_{rcv}}^{+}(z : q_n) B_{ml, N_{rcv}}^{+} + A_{N_{rcv}}^{-} F_{N_{rcv}}^{-}(z : q_n) B_{ml, N_{rcv}}^{-}\} \{ {\cos(m'\varphi), \sigma' = e \atop \sin(m'\varphi), \sigma' = o} \} \end{gathered} \tag{25.4}$$

Here φ_s is the angle between the x-axis of the local coordinate system at the source and the location of the receiver as seen from the source, and φ is the angle between the x-axis of the local coordinate system at the receiver and the location of the source as seen from the receiver.

In Section 20, the following expansion of the directivity functions of the source and receiver in terms of spherical harmonics was introduced:

$$d_{Src,\sigma ml} = \int_0^{2\pi} d\varphi \int_0^{\pi} \sin(\vartheta) d\vartheta Y_{\sigma ml}(\vartheta,\varphi) D_{Src}(\vartheta,\varphi) \tag{25.5}$$

$$d_{Rcv,\sigma ml} = \int_0^{2\pi} d\varphi \int_0^{\pi} \sin(\vartheta) d\vartheta Y_{\sigma ml}(\vartheta,\varphi) D_{Rcv}(\vartheta,\varphi) \tag{25.6}$$

Neglecting multiple scattering terms, the normal mode contribution of the scattering from an elastic target has the form:

$$\begin{aligned} &\psi_{Scattered}(q:q') = \\ &\sum_{\sigma ml}\sum_{\sigma'm'l'}\sum_{\sigma''m''l''}\sum_{\sigma'''m'''l'''} d_{Src,\sigma ml}\Gamma_{\sigma ml,N_{Src};\sigma''m''l'',N_{Tgt}}(r_{Src-Tgt}, z_{Src}, z_{Tgt}, \varphi_{Src-Tgt}, \varphi_{Tgt-Src} : q) \\ &T_{\sigma'm'l',\sigma''m''l''}\Gamma_{\sigma'm'l',N_{Tgt};\sigma'''m'''l''',N_{Rcv}}(r_{Rcv-Tgt}, z_{Tgt}, z_{Rcv}, \varphi_{Tgt-Rcv}, \varphi_{Rcv-Tgt} : q') d_{Rcv,\sigma'''m'''l'''} \end{aligned} \tag{25.7}$$

The addition of multiple scattering terms to the calculations replaces the free field T-matrix in the above expression with the waveguide T-matrix of the following form:

$$T^{W} = (T^{-1} + i\widetilde{R})^{-1} \tag{25.8}$$

Here, the matrix $\widetilde{R}$ is the rescattering matrix of References 10 through 12. Generally, the rescattering matrix only affects the scattered field by a few decibels.

26. NORMAL MODE REPRESENTATION OF SCATTERING FROM A ROUGH INTERFACE

This section presents the normal mode description of the scattering from a rough interface. This formalism can be utilized to describe both surface and bottom reverberation from a rough interface. The topic of bottom reverberation due to volume inhomogeneities in the sediment will be addressed in a separate section on the normal mode representation of volume reverberation.

This section begins by recalling Equation 19.29 representing the expansion of the normal mode contribution to the Greens' Function in terms of a spherical basis about the source:

$$A_{\sigma ml,N_{src}}(r,z_s,z,\varphi_s:q_n)=\frac{2\pi i}{\rho(z_s)}\sqrt{\frac{\varepsilon(m)}{8\pi}}H_m^{(1)}(q_n r)F_{N_{rcv}}(z:q_n)\left\{\begin{matrix}\cos(m\varphi_s),\sigma=e\\ \sin(m\varphi_s),\sigma=o\end{matrix}\right\}$$
$$\{A^+_{N_{src}}F^+_{N_{src}}(z_s:q_n)B^+_{ml,N_{src}}+A^-_{N_{src}}F^-_{N_{src}}(z_s:q_n)B^-_{ml,N_{src}}\} \tag{26.1}$$

Expand the normal mode contribution to the Greens' Function in terms of a cylindrical basis set about the field point:

$$A_{\sigma ml,N_{src}}(|\vec{r}+\Delta\vec{r}|,z_s,z+\Delta z,\varphi_s:q_n)=$$
$$\sum_{s=+}^{-}\sum_{\sigma'm'}B^s{}_{\sigma ml,N_{src};\sigma'm',N_{Rcv}}(r,z_s,z,\varphi_s,\varphi_r:q_n)\,\mathrm{Re}\,\chi^s_{\sigma'm'}(\Delta r,\Delta z,\Delta\varphi) \tag{26.2}$$

One may make use of the following expansion of the Hankel Function:

$$H_m^{(1)}(w)=\sum_{m'=0}^{\infty}\frac{\varepsilon(m')}{2}\{H^{(1)}_{m+m'}(u)+(-1)^{m'}H^{(1)}_{m-m'}(u)\}J_{m'}(u')\cos(m'\varphi) \tag{26.3}$$

$$w=\sqrt{u^2+u'^2-2uu'\cos(\varphi)} \tag{26.4}$$

One may also make use of the following representation of the depth function in terms of upward and downward-going plane waves:

$$F_N(z+\Delta z:q)=A_N^+(q)F_N^+(z:q)\exp(+ih_N\Delta z)+A_N^-(q)F_N(z:q)\exp(-ih_N\Delta z) \tag{26.5}$$

Substituting Equations 26.3 and 26.5 into Equation 26.1, one obtains the following expression for the expansion coefficients in Equation 26.2.

$$B^{s}{}_{\sigma ml,N_{src}:\sigma' m',N}(r,z_s,z,\varphi_s,\varphi_r:q_n) =$$
$$\frac{\pi i}{\rho(z_s)}\sqrt{\varepsilon(m)\varepsilon(m')}\{H^{(1)}_{m+m'}(q_n r)+(-1)^{m'}H^{(1)}_{m-m'}(q_n r)\}$$
$$\{A^{+}_{N_{src}}F^{+}_{N_{src}}(z_s:q_n)B^{+}_{ml,N_{src}}+A^{-}_{N_{src}}F^{-}_{N_{src}}(z_s:q_n)B^{-}_{ml,N_{src}}\}A^{s}_{N_{rcv}}F^{s}_{N_{rcv}}(z:q_n) \tag{26.6}$$
$$\{\begin{matrix}\cos(m\varphi_s),\sigma=e\\ \sin(m\varphi_s),\sigma=o\end{matrix}\}\{\begin{matrix}\cos(m\varphi_r),\sigma'=e\\ \sin(m\varphi_r),\sigma'=o\end{matrix}\}$$

Here φ_s is the angle between the x-axis of the local coordinate system at the source and the location of the receiver as seen from the source, and φ_r is the angle between the x-axis of the local coordinate system at the receiver and the location of the source as seen from the receiver.

One may introduce the following expansion of the directivity functions of the source and receiver in terms of spherical harmonics:

$$d_{Src,\sigma ml} = \int_0^{2\pi} d\varphi \int_0^{\pi} \sin(\vartheta)d\vartheta Y_{\sigma ml}(\vartheta,\varphi)D_{Src}(\vartheta,\varphi) \tag{26.7}$$

$$d_{Rcv,\sigma ml} = \int_0^{2\pi} d\varphi \int_0^{\pi} \sin(\vartheta)d\vartheta Y_{\sigma ml}(\vartheta,\varphi)D_{Rcv}(\vartheta,\varphi) \tag{26.8}$$

The above expansion is used to include the effects of the beam pattern of the source and receiver on the scattered field.

Suppose scattering from the interface is described statistically by a scattering function $S(\vartheta,\vartheta',\varphi)$ dependent upon the frequency and the incident and scattered angles. The scattering from the rough interface is equal to the surface integral of the product of the incident field, the scattered field, and the scattering function over the rough surface. Decompose the rough interface into rectangular facets, whose areas are small in comparison to the range to and from the facet. In this case, the scattering function is approximately constant over the area of the facet, and the surface integral over the facet requires the integration of the following product of cylindrical wave functions:

$$\int_{-l/2}^{+l/2} dx \int_{-h/2}^{+h/2} dy \operatorname{Re}\chi^{s}_{\sigma m}(r,0,\varphi:q)\operatorname{Re}\chi^{s'}_{\sigma' m'}(r,0,\varphi:q') \tag{26.9}$$

In general, the above integral is restricted to one over either the upward or downward-going components of the cylindrical wave functions. Equation 26.9 may be expressed as follows:

$$S_{\sigma m,\sigma' m'}(q:q') = \frac{\sqrt{\varepsilon(m)\varepsilon(m')}}{8\pi}\int_{-l/2}^{+l/2} dx \int_{-h/2}^{+h/2} dy J_m(rq)J_{m'}(rq')\{\begin{matrix}\cos(m\varphi),\sigma=e\\ \sin(m\varphi),\sigma=o\end{matrix}\}\{\begin{matrix}\cos(m'\varphi),\sigma'=e\\ \sin(m'\varphi),\sigma'=o\end{matrix}\} \tag{26.10}$$

One may simplify the calculation by replacing the rectangular facet by a circular facet of the same area. In this case, Equation 26.10 takes on the following simplified form:

$$S_{\sigma m,\sigma' m'}(q:q') \approx \frac{1}{4}\delta_{\sigma}^{\sigma'}\delta_{m}^{m'}\int_0^R rdrJ_m(qr)J_m(q'r) \tag{26.11}$$

In the limit $R \to +\infty$, the integral approaches the following limit:

$$\lim_{R\to+\infty} S_{\sigma m,\sigma' m'}(q:q') = \frac{1}{4q}\delta_{\sigma}^{\sigma'}\delta_{m}^{m'}\delta(q-q') \tag{26.12}$$

Here, one has used the fact the following integral is equal to the Dirac delta distribution:

$$\int_0^\infty xdxJ_m(qx)J_m(q'x) = \frac{1}{q}\delta(q-q') \tag{26.13}$$

In order to simplify Equation 26.11 and obtain a closed form solution, replace the above integral by the following Gaussian weighted integral:

$$S_{\sigma m,\sigma' m'}(q:q') \approx \frac{1}{4}\delta_{\sigma}^{\sigma'}\delta_{m}^{m'}\int_0^{+\infty} rdrJ_m(qr)J_m(q'r)\exp(-a^2r^2) \tag{26.14}$$

The parameter "a" is chosen such that the following integral is equal to the area of the facet:

$$2\pi\int_0^\infty rdr\exp(-a^2r^2) = \frac{1}{2}a^{-2} = \pi R^2 = lh \tag{26.15}$$

$$a = \frac{1}{\sqrt{2\pi R^2}} = \frac{1}{\sqrt{2lh}} \tag{26.16}$$

One may regard this approximation as replacing the incident field by a Gaussian beam with the appropriate beam width.

The following integral is found on page 718 of Reference 18:

$$\int_0^\infty xdx\exp(-a^2x^2)J_m(\alpha x)J_m(\beta x) = \frac{1}{2a^2}\exp(-\frac{\alpha^2+\beta^2}{4a^2})I_m(\frac{\alpha\beta}{2a^2}) \tag{26.17}$$

Substituting this integral into Equation 26.14, one arrives at the following approximation of Equation 26.11.

$$S_{\sigma m,\sigma' m'}(q:q') = \frac{\pi R^2}{4}\delta_{\sigma}^{\sigma'}\delta_{m}^{m'}\exp(-(q^2+q'^2)\pi R^2/2)I_m(qq'\pi R^2) \tag{26.18}$$

The integral given in Equation 26.18 is proportional to the area of the facet. The ratio of Equation 26.18 and the area of the facet may be regarded as the directivity function of a flat facet. The normal mode contribution to the scattering from the facet is given by the following expression:

$$\psi_{Scattered}(\vec{r}_{sb},\vec{r}_{rb}:q,q')=\frac{\rho(z_r)}{\rho(z_b)}\sum_{\sigma ml}\sum_{s=+}^{-}\sum_{\sigma'm'}\sum_{\sigma''m''}\sum_{s'=+}^{-}\sum_{\sigma'''m'''l''} d_{Src,\sigma ml}B^{s}_{\sigma ml,N_{Src};\sigma'm',N_b}(r_{sb},z_s,z_b,\varphi_{sb},\varphi_{bs}:q)$$
$$S^{s,s'}_{\sigma'm',\sigma''m''}(\alpha,\alpha',\beta:q,q')B^{s'}_{\sigma'''m'''l''',N_{Rcv};\sigma''m'',N_b}(r_{br},z_r,z_b,\varphi_{rb},\varphi_{br}:q')d_{Rcv,\sigma'''m'''l'''} \tag{26.19}$$

where r_{sb} is the horizontal range from the source to the facet, r_{br} is the horizontal range from the facet to the receiver, z_s is the depth of the source, z_r is the depth of the receiver, and z_b is the depth of the facet. Other variables are the angle between the x-axis at the source and the location of the facet (φ_{sb}), the angle between the x-axis at the facet and the location of the source (φ_{bs}), the angle between the x-axis at the receiver and the location of the facet (φ_{rb}), and the angle between the x-axis at the facet and the location of the receiver (φ_{br}).

The function $S^{s,s'}_{\sigma'm',\sigma''m''}(\alpha,\alpha',\beta:q,q')$ is given by the following expression in terms of the scattering function $S(\vartheta,\vartheta',\varphi)$ and the function $S_{\sigma m,\sigma'm'}(q:q')$ defined in Equation 26.18.

$$S^{s,s'}_{\sigma m,\sigma'm'}(\alpha,\alpha',\beta:q,q')=S_{\sigma m,\sigma'm'}(q:q')\begin{cases}S(+\alpha,-\alpha',\beta), s=+,s'=+\\ S(+\alpha,+\alpha',\beta), s=+,s'=-\\ S(-\alpha,-\alpha',\beta), s=-,s'=+\\ S(-\alpha,+\alpha',\beta), s=-,s'=-\end{cases} \tag{26.20}$$

The angles α, α' are the grazing angles of the incident and scattered field at the interface defined by the following relationship, and the angle β is the azimuthal angle between the source and receiver as seen from the facet:

$$q=k_{N_b}\cos(\alpha) \tag{26.21a}$$

$$q'=k_{N_b}\cos(\alpha') \tag{26.21b}$$

In the case one is given a statistical representation of the scattering from a randomly rough facet in terms of a scattering strength, Equation 26.20 is replaced by the following function, where the function $S(+\alpha,+\alpha',\beta)$ is the square root of the scattering strength:

$$S^{s,s'}_{\sigma m,\sigma' m'}(\alpha,\alpha',\beta:q,q') = S^{Random}_{\sigma m,\sigma' m'}(q:q')\begin{cases} S(+\alpha,-\alpha',\beta), s=+, s'=+ \\ S(+\alpha,+\alpha',\beta), s=+, s'=- \\ S(-\alpha,-\alpha',\beta), s=-, s'=+ \\ S(-\alpha,+\alpha',\beta), s=-, s'=- \end{cases} \tag{26.22}$$

$$S^{Random}_{\sigma m,\sigma' m'}(q:q') = \frac{\sqrt{\pi R^2}}{4}\delta^{\sigma'}_{\sigma}\delta^{m'}_{m}\exp(-(q^2+q'^2)\pi R^2/2)I_m(qq'\pi R^2)\sqrt{-2\log(u_1)}\exp(+2\pi i u_2) \tag{26.23}$$

Here u_1 and u_2 are a pair of uniformly distributed random numbers in the interval from 0 to 1 used to generate a single realization of a randomly rough surface.

27. NORMAL MODE REPRESENTATION OF SCATTERING FROM VOLUME INHOMOGENEITIES

This section presents the normal mode description of the scattering from volume inhomogeneities. This formalism can be utilized to describe scattering from volume inhomogeneities in the case of both volume and bottom reverberation. For simplicity, attention will be restricted to the case of isotropic scattering from volume inhomogeneities.

This section begins by recalling Equation 19.29 representing the expansion of the normal mode contribution to the Greens' Function in terms of a spherical basis about the source:

$$A_{\sigma ml,N_{src}}(r,z_s,z,\varphi_s:q_n)=\frac{2\pi i}{\rho(z_s)}\sqrt{\frac{\varepsilon(m)}{8\pi}}H_m^{(1)}(q_n r)F_{N_{rcv}}(z:q_n)\{\begin{matrix}\cos(m\varphi_s),\sigma=e\\ \sin(m\varphi_s),\sigma=o\end{matrix}\}$$
$$\{A^+_{N_{src}}F^+_{N_{src}}(z_s:q_n)B^+_{ml,N_{src}}+A^-_{N_{src}}F^-_{N_{src}}(z_s:q_n)B^-_{ml,N_{src}}\} \tag{27.1}$$

Express Equation 27.1 in terms of upward and downward-going plane waves about the field point:

$$A_{\sigma ml,N_{src}}(r,z_s,z,\varphi_s:q_n)=$$
$$A^+{}_{\sigma ml,N_{src},N}(r,z_s,\varphi_s:q_n)\exp(+ih_N(z-z_N))+A^-{}_{\sigma ml,N_{src}}(r,z_s,\varphi_s:q_n)\exp(-ih_N(z-z_N)) \tag{27.2}$$

$$A^\pm{}_{\sigma ml,N_{src},N}(r,z_s,\varphi_s:q_n)=\frac{2\pi i}{\rho(z_s)}\sqrt{\frac{\varepsilon(m)}{8\pi}}H_m^{(1)}(q_n r)\{\begin{matrix}\cos(m\varphi_s),\sigma=e\\ \sin(m\varphi_s),\sigma=o\end{matrix}\}$$
$$\{A^+_{N_{src}}F^+_{N_{src}}(z_s:q_n)B^+_{ml,N_{src}}+A^-_{N_{src}}F^-_{N_{src}}(z_s:q_n)B^-_{ml,N_{src}}\}A^\pm{}_N(q_n) \tag{27.3}$$

Here, one makes use of the following expansion of the depth function in terms of the propagator matrix:

$$F_N(z:q)=A^+_N(q)\exp(+ih_N(z-z_N))+A^-_N(q)\exp(-ih_N(z-z_N)) \tag{27.4}$$

Introduce the following expansion of the directivity functions of the source and receiver in terms of spherical harmonics:

$$d_{Src,\sigma ml} = \int_0^{2\pi} d\varphi \int_0^{\pi} \sin(\vartheta) d\vartheta Y_{\sigma ml}(\vartheta,\varphi) D_{Src}(\vartheta,\varphi) \tag{27.5}$$

$$d_{Rcv,\sigma ml} = \int_0^{2\pi} d\varphi \int_0^{\pi} \sin(\vartheta) d\vartheta Y_{\sigma ml}(\vartheta,\varphi) D_{Rcv}(\vartheta,\varphi) \tag{27.6}$$

Use the above expansion to include the effects of the beam pattern of the source and receiver on the scattered field.

Consider the isotropic scattering due to volume inhomogeneities from a given volume element whose horizontal cross section is constant. Scattering from such volume inhomogeneities can be approximated by the following integral over depth:

$$\sum_{s=+}^{-}\sum_{s'=+}^{-}\sum_{\sigma ml} d_{Src,\sigma ml} \sum_{\sigma' m' l'} d_{Rcv,\sigma' m' l'} Area \int_{z}^{z+d} dz \frac{\rho(z_r)}{\rho(z)} S_{Volume} A^{s}_{\sigma ml,N_{Src}}(r_s,z_s,z,\varphi_s:q) A^{s'}_{\sigma' m' l',N_{Rcv}}(r_r,z_r,z,\varphi_r:q') \tag{27.7}$$

Here, $Area$ is the area of the horizontal cross-section of the volume element and S_{Volume} is the linear scattering strength per unit volume, that is, the scattered field at unit distance is equal to the product of the linear scattering strength, the volume of the scattering volume, and the incident field. Assume the scattering strength is constant and isotropic within each layer of the waveguide. In this case, the integral over depth in each of the layers in the waveguide contains an integral of the following form:

$$\begin{aligned} &\int_{z_N}^{z_{N+1}} dz \exp[+ish_N(z-z_N)+is'h'_N(z-z_N)] = \\ &\frac{1}{i(sh_N+s'h'_N)}\{\exp[+ish_N(z_{N+1}-z_N)+is'h'_N(z_{N+1}-z_N)]-1\} \end{aligned} \tag{27.8}$$

In the case the denominator $i(sh+s'h')$ vanishes, this integral is equal to the depth of the layer $d_N=(z_{N+1}-z_N)$. The depth integrated scattering from volume inhomogeneities with the given horizontal cross-sectional area may therefore be expressed in terms of the following summation:

$$\begin{aligned} &\sum_{N}\sum_{s=+}^{-}\sum_{s'=+}^{-}\sum_{\sigma ml}\sum_{\sigma' m' l'} d_{Src,\sigma ml} d_{Rcv,\sigma' m' l'} \frac{\{\exp[(+ish_N+is'h'_N)(z_{N+1}-z_N)]-1\}}{(+ish_N+is'h'_N)} \\ &\frac{\rho(z_r)}{\rho_N} S_{Volume,N} Area A^{s}_{\sigma ml,N_{Src},N}(r_s,z_s,\varphi_s:q) A^{s'}_{\sigma' m' l',N_{Rcv},N}(r_r,z_r,\varphi_r:q') \end{aligned} \tag{27.9}$$

Here, ρ_N is the density of the N'th layer, and $S_{Volume,N}$ is the linear scattering strength of the N'th layer.

Consider the case were the scattering from a random collection of volume inhomogeneities is described by a depth dependent volume scattering strength. The energy in the scattered field at a unit distance is equal to the product of the volume scattering strength, the volume of the volume containing the scatterers, and the energy of the incident signal. In this case, replace the product of the linear scattering strength and the cross-sectional area of the volume in Equation 27.9 by the following expression:

$$S_{Volume,N} Area \rightarrow \sqrt{\frac{VSS_N Area}{d_N}} \sqrt{-2\log(u_1)} \exp(+2\pi i u_2) \tag{27.10}$$

Here, VSS_N is the volume scattering strength in the N'th layer, d_N is the thickness of the N'th layer, and u_1 and u_2 are a pair of uniformly distributed random variables between 0 and 1. In this manner, one produces a single realization of the scattering from random volume inhomogeneities. Equation 27.10 describes a zero mean Gaussian random process, with standard deviation squared proportional the product of the volume and the volume scattering strength.

28. TIME DOMAIN REPRESENTATION OF SCATTERING FROM A ROUGH INTERFACE AND VOLUME INHOMOGENEITIES

This section combines the results of Sections 15, 26 and 27 to create a time domain representation of the normal mode contribution to the scattering from a rough interface and volume inhomogeneities. Our attention will be limited to the propagation of a simple Gaussian pulse with the following frequency spectrum:

$$S(\omega) = \exp[-\frac{(\omega-\omega_0)^2}{2\Delta\omega^2}] \tag{28.1}$$

Here, $\omega_0 = 2\pi f_0$ is the angular frequency of the center frequency, and $\Delta\omega$ is equal to 2π times the bandwidth of the incident signal. The time domain representation of the incident signal is of the following form:

$$s(t) = \int_{-\infty}^{+\infty} d\omega S(\omega)\exp[-i\omega t] = \sqrt{2\pi}\Delta\omega\exp[-t^2\Delta\omega^2/2 - i\omega_0 t] \tag{28.2}$$

The approach adopted in this section is to approximate the scattered field by a Gaussian integral of the following form:

$$s_{Scattered}(t) = \int_{-\infty}^{+\infty} d\omega F(\omega)\exp[-\alpha^2(\omega-\omega_0)^2 + \beta(\omega-\omega_0)] \tag{28.3}$$

One approximates this integral by either the saddle point approximation or the method of stationary phase. In the case of a broadband signal, the bandwidth is sub-divided into a collection of sub-bands for which the above technique is applied:

$$s_{Scattered}(t) \approx F(\omega_0)\int_{-\infty}^{+\infty} d\omega\exp[-\alpha^2(\omega-\omega_0)^2 + \beta(\omega-\omega_0)] = F(\omega_0)\frac{\sqrt{\pi}}{\alpha}\exp[+(\beta/2\alpha)^2] \tag{28.4}$$

This technique is extended to an arbitrary band-limited signal by convolving the scattered signal from a broadband Gaussian pulse with the incident signal. This technique approximates the spectrum of the incident signal by the following Gaussian weighted spectrum, where the bandwidth of the Gaussian pulse is sufficiently large this approximation has adequate accuracy:

$$S(\omega) \approx S(\omega)\exp[-(\omega-\omega_0)^2/2\Delta\omega^2] \tag{28.5}$$

From Equations 26.6, 26.18, 26.19, and 26.20 one obtains the following representation of the scattering of a monochromatic wave by a facet on a rough interface, where the function $S(\vartheta,\vartheta',\varphi)$ is the bistatic scattering function of the rough surface:

$$\psi_{Scattered}(\vec{r}_{sb},\vec{r}_{rb} : q,q') =$$
$$\frac{\rho(z_r)}{\rho(z_b)}\sum_{\sigma ml}\sum_{s=+}^{-}\sum_{\sigma'm'}\sum_{\sigma''m''}\sum_{s'=+}^{-}\sum_{\sigma'''m'''l'''} d_{Src,\sigma ml}B^{s}_{\sigma ml,N_{Src};\sigma'm',N_b}(r_{sb},z_s,z_b,\varphi_{sb},\varphi_{bs} : q) \tag{28.6}$$
$$S^{s,s'}_{\sigma'm',\sigma''m''}(\alpha,\alpha',\beta : q,q')B^{s'}_{\sigma'''m'''l''',N_{Rcv};\sigma''m'',N_b}(r_{br},z_r,z_b,\varphi_{rb},\varphi_{br} : q')d_{Rcv,\sigma'''m'''l'''}$$

$$B^{s}{}_{\sigma ml,N_{src};\sigma'm',N}(r,z_s,z,\varphi_s,\varphi_r : q_n) =$$
$$\frac{\pi i}{\rho(z_s)}\sqrt{\varepsilon(m)\varepsilon(m')}\{H^{(1)}_{m+m'}(q_n r)+(-1)^{m'}H^{(1)}_{m-m'}(q_n r)\}$$
$$\{A^{+}_{N_{src}}F^{+}_{N_{src}}(z_s : q_n)B^{+}_{ml,N_{src}} + A^{-}_{N_{src}}F^{-}_{N_{src}}(z_s : q_n)B^{-}_{ml,N_{src}}\}A^{s}_{N_{rcv}}F^{s}_{N_{rcv}}(z : q_n) \tag{28.7}$$
$$\left\{\begin{matrix}\cos(m\varphi_s),\sigma = e\\ \sin(m\varphi_s),\sigma = o\end{matrix}\right\}\left\{\begin{matrix}\cos(m\varphi_r),\sigma' = e\\ \sin(m\varphi_r),\sigma' = o\end{matrix}\right\}$$

$$S^{s,s'}_{\cdot m,\sigma'm'}(\alpha,\alpha',\beta : q,q') = S_{\sigma m,\sigma'm'}(q : q')\begin{cases}S(+\alpha,-\alpha',\beta), s = +, s' = +\\ S(+\alpha,+\alpha',\beta), s = +, s' = -\\ S(-\alpha,-\alpha',\beta), s = -, s' = +\\ S(-\alpha,+\alpha',\beta), s = -, s' = -\end{cases} \tag{28.8}$$

$$S_{\sigma m,\sigma'm'}(q : q') = \frac{\pi R^2}{4}\delta^{\sigma'}_{\sigma}\delta^{m'}_{m}\exp(-(q^2+q'^2)\pi R^2/2)I_m(qq'\pi R^2) \tag{28.9}$$

The problem of deriving the time domain representation of the scattering from a rough surface entails expressing the following integral in the form of Equation 28.3:

$$\psi_{Scattered}(\vec{r}_{sb},\vec{r}_{rb},: q_n,q_{n'} : t) =$$
$$\int_{-\infty}^{+\infty} d\omega \exp[-i\omega t]\exp[-(\omega-\omega_0)^2/2\Delta\omega^2]\psi_{Scattered}(\vec{r}_{sb},\vec{r}_{rb},: q_n,q_{n'} : \omega) \tag{28.10}$$

One may begin by deriving an asymptotic expression for the coefficients defined in Equation 28.7 by substituting the following asymptotic expression for the Hankel Functions:

$$H^{(1)}_{m}(x) \approx \sqrt{\frac{2}{\pi x}}\exp[ix - im\pi/2 - i\pi/4] \tag{28.11}$$

Upon substitution of Equation 28.11 into Equation 28.7, one obtains the following asymptotic expression:

$$B^{s}_{\sigma ml,N_{Src};\sigma'm',N}(r,z_s,z,\varphi_s,\varphi_r:q_n)\approx \sqrt{\frac{2}{\pi q_n r}}\exp[iq_n r-i\pi/4]\widetilde{B}^{s}_{\sigma ml,N_{Src};\sigma'm',N}(z_s,z,\varphi_s,\varphi_r:q_n) \tag{28.12}$$

$$\widetilde{B}^{s}{}_{\sigma ml,N_{src};\sigma'm',N_{rcv}}(z_s,z,\varphi_s,\varphi_r:q_n)=\frac{2\pi i}{\rho(z_s)}(-i)^{m+m'}\sqrt{\varepsilon(m)\varepsilon(m')}$$
$$\{A^{+}_{N_{src}}F^{+}_{N_{src}}(z_s:q_n)B^{+}_{ml,N_{src}}+A^{-}_{N_{src}}F^{-}_{N_{src}}(z_s:q_n)B^{-}_{ml,N_{src}}\}A^{s}_{N_{rcv}}F^{s}_{N_{rcv}}(z:q_n) \tag{28.13}$$
$$\left\{\begin{matrix}\cos(m\varphi_s),\sigma=e\\ \sin(m\varphi_s),\sigma=o\end{matrix}\right\}\left\{\begin{matrix}\cos(m\varphi_r),\sigma'=e\\ \sin(m\varphi_r),\sigma'=o\end{matrix}\right\}$$

Next, one derives an asymptotic expression for the directivity function of the facet based on the asymptotic expansion for the modified Bessel Function in Equation 28.9. Make the following approximation for Equation 28.9:

$$S_{\sigma m,\sigma'm'}(q:q')\approx\frac{\pi R^2}{4}\delta^{\sigma'}_{\sigma}\delta^{m'}_{m}\exp[-(q-q')^2\pi R^2/2]\exp[-qq'\pi R^2]I_m(qq'\pi R^2) \tag{28.14}$$

Define the following asymptotic expansion of Equation 28.8:

$$S^{s,s'}_{'m,\sigma'm'}(\alpha,\alpha',\beta:q,q')\approx\exp[-(q-q')^2\pi R^2/2]\widetilde{S}^{s,s'}_{'m,\sigma'm'}(\alpha,\alpha',\beta:q,q') \tag{28.15}$$

$$\widetilde{S}^{s,s'}_{'m,\sigma'm'}(\alpha,\alpha',\beta:q,q')=$$
$$\frac{\pi R^2}{4}\delta^{\sigma'}_{\sigma}\delta^{m'}_{m}\exp[-qq'\pi R^2]I_m(qq'\pi R^2)\begin{cases}S(+\alpha,-\alpha',\beta),s=+,s'=+\\ S(+\alpha,+\alpha',\beta),s=+,s'=-\\ S(-\alpha,-\alpha',\beta),s=-,s'=+\\ S(-\alpha,+\alpha',\beta),s=-,s'=-\end{cases} \tag{28.16}$$

Substituting Equations 28.12 and 28.15 into Equation 28.6, one obtains the following asymptotic expansion for the scattering from a facet on a rough interface:

$$\psi_{Scattered}(\vec{r}_{sb},\vec{r}_{rb}:q,q')=$$
$$\frac{2}{\pi i}\sqrt{\frac{1}{qq'r_{sb}r_{rb}}}\exp[i(qr_{sb}+q'r_{rb})-(q-q')^2\pi R^2/2]\widetilde{\psi}_{Scattered}(\vec{r}_{sb},\vec{r}_{rb}:q,q') \tag{28.17}$$

$$\widetilde{\psi}_{Scattered}(\vec{r}_{sb},\vec{r}_{rb}:q,q')=$$
$$\frac{\rho(z_r)}{\rho(z_b)}\sum_{\sigma ml}\sum_{s=+}^{-}\sum_{\sigma'm'}\sum_{\sigma''m''}\sum_{s'=+}^{-}\sum_{\sigma'''m'''l'''}d_{Src,\sigma ml}\widetilde{B}^{s}_{\sigma ml,N_{Src};\sigma'm',N_b}(z_s,z_b,\varphi_{sb},\varphi_{bs}:q) \tag{28.18}$$
$$\widetilde{S}^{s,s'}_{\sigma'm',\sigma''m''}(\alpha,\alpha',\beta:q,q')\widetilde{B}^{s'}_{\sigma'''m'''l''',N_{Rcv};\sigma''m'',N_b}(z_r,z_b,\varphi_{rb},\varphi_{br}:q')d_{Rcv,\sigma'''m'''l'''}$$

Substituting Equation 28.17 for the asymptotic expression for the scattered field from a monochromatic source into Equation 28.10 one obtains the following expression:

$$\psi_{Scattered}(\vec{r}_{sb},\vec{r}_{rb},: q_n, q_{n'} : t) =$$
$$\int_{-\infty}^{+\infty} d\omega \exp[-i\omega t]\exp[-(\omega-\omega_0)^2/2\Delta\omega^2 + i(q_n r_{sb} + q_{n'} r_{rb}) - (q_n - q_{n'})^2 \pi R^2/2] \tag{28.19}$$
$$\frac{2}{\pi i}\sqrt{\frac{1}{q_n q_{n'} r_{sb} r_{rb}}}\tilde{\psi}_{Scattered}(\vec{r}_{sb},\vec{r}_{rb}, q_n, q_{n'} : \omega)$$

Expand the integral about the center frequency by making the following substitutions and keeping only terms of quadratic order in the exponential evaluated about the center frequency:

$$q_n \approx q_{n0} + q'_{n0}(\omega-\omega_0) + q''_{n0}(\omega-\omega_0)^2/2 \tag{28.20a}$$

$$q_{n'} \approx q_{n'0} + q'_{n'0}(\omega-\omega_0) + q''_{n'0}(\omega-\omega_0)^2/2 \tag{28.20b}$$

$$q_{n0} = q_n(\omega_0) \tag{28.20c}$$

$$q'_{n0} = \frac{dq_n(\omega=\omega_0)}{d\omega} \tag{28.20d}$$

$$q''_{n0} = \frac{d^2 q_n(\omega=\omega_0)}{d\omega^2} \tag{28.20e}$$

$$\psi_{Scattered}(\vec{r}_{sb},\vec{r}_{rb},: q_n, q_{n'} : t) =$$
$$\frac{2}{\pi i}\sqrt{\frac{1}{q_{n0} q_{n0'} r_{sb} r_{rb}}}\tilde{\psi}_{Scattered}(\vec{r}_{sb},\vec{r}_{rb}, q_{n0}, q_{n'0} : \omega_0) \tag{28.21}$$
$$\exp[i(q_{n0} r_{sb} + q_{n'0} r_{rb}) - i\omega_0 t - (q_{n0} - q_{n'0})^2 \pi R^2/2]\int_{-\infty}^{+\infty} d\omega \exp[-\alpha^2(\omega-\omega_0)^2 - i\beta(\omega-\omega_0)]$$

$$\alpha^2 = \frac{1}{2\Delta\omega^2} - i(q''_{n0} r_{sb} + q''_{n'0} r_{rb}) + (q'_{n0} - q'_{n'0})^2 \pi R^2/2 \tag{28.22}$$

$$\beta = t - (q'_{n0} r_{sb} + q'_{n'0} r_{rb}) - i(q_{n0} - q_{n'0})(q'_{n0} - q'_{n'0})\pi R^2 \tag{28.23}$$

Upon completing the Gaussian in Equation 28.21, one arrives at the following expression for the field scattered from the facet on the rough interface:

$$\psi_{Scattered}(\vec{r}_{sb},\vec{r}_{rb},:q_n,q_{n'}:t)=$$
$$\frac{2}{\pi i}\sqrt{\frac{1}{q_{n0}q_{n0'}r_{sb}r_{rb}}}\tilde{\psi}_{Scattered}(\vec{r}_{sb},\vec{r}_{rb},q_{n0},q_{n'0}:\omega_0) \tag{28.24}$$
$$\exp[i(q_{n0}r_{sb}+q_{n'0}r_{rb})-i\omega_0 t_0-(q_{n0}-q_{n'0})^2\pi R^2/2]\frac{\sqrt{2\pi}}{\tau}\exp[-(t-t_0)^2/2\tau^2-i\omega_0(t-t_0)]$$

$$t_0=(q'_{n0}\,r_{sb}+q'_{n'0}\,r_{rb})-i(q_{n0}-q_{n'0})(q'_{n0}-q'_{n'0})\pi R^2 \tag{28.25}$$

$$\tau^2=2\alpha^2=\frac{1}{\Delta\omega^2}-2i(q''_{n0}\,r_{sb}+q''_{n'0}\,r_{rb})+(q'_{n0}-q'_{n'0})^2\pi R^2 \tag{28.26}$$

The parameters t_0 and τ are the complex arrival time and pulse-length of the scattered signal, respectively. Neglecting terms proportional to the area of the facet, the travel time reduces to the sum of the travel times of the two modes.

From Equations 27.3 and 27.9, one obtains the following expression for the depth integrated scattering from volume inhomogeneities with horizontal cross-sectional area $Area$:

$$\psi_{Scattered}(r_s,r_r,z_s,z_r,\varphi_s,\varphi_r:q_n,q_{n'}:\omega)=$$
$$\sum_N\sum_{s=+}^{-}\sum_{s'=+}^{-}\sum_{\sigma ml}\sum_{\sigma'm'l'}d_{Src,\sigma ml}d_{Rcv,\sigma'm'l'}\frac{\{\exp[(+ish_N+is'h'_N)(z_{N+1}-z_N)]-1\}}{(+ish_N+is'h'_N)} \tag{28.27}$$
$$\frac{\rho(z_r)}{\rho_N}S_{Volume,N}AreaA^{s}_{\sigma ml,N_{Src},N}(r_s,z_s,\varphi_s:q_n)A^{s'}_{\sigma'm'l',N_{Rcv},N}(r_r,z_r,\varphi_r:q_{n'})$$

$$A^{\pm}{}_{\sigma ml,N_{src},N}(r,z_s,\varphi_s:q_n)=\frac{2\pi i}{\rho(z_s)}\sqrt{\frac{\varepsilon(m)}{8\pi}}H_m^{(1)}(q_nr)\{\begin{matrix}\cos(m\varphi_s),\sigma=e\\ \sin(m\varphi_s),\sigma=o\end{matrix}\} \tag{28.28}$$
$$\{A^+_{N_{src}}F^+_{N_{src}}(z_s:q_n)B^+_{ml,N_{src}}+A^-_{N_{src}}F^-_{N_{src}}(z_s:q_n)B^-_{ml,N_{src}}\}A^{\pm}{}_N(q_n)$$

The time domain representation of the scattered field is given by the following equation:

$$\psi_{Scattered}(r_s,r_r,,z_s,z_r,\varphi_s,\varphi_r:q_n,q_{n'}:t)=$$
$$\int_{-\infty}^{+\infty}d\omega\exp[-i\omega t]\exp[-(\omega-\omega_0)^2/2\Delta\omega^2]\psi_{Scattered}(r_s,r_r,,z_s,z_r,\varphi_s,\varphi_r:q_n,q_{n'}:\omega) \tag{28.29}$$

Substituting the asymptotic expansion for the Hankel Function into Equation 28.28, one arrives at the following asymptotic expansion of Equation 28.27:

$$\psi_{Scattered}(r_s,r_r,z_s,z_r,\varphi_s,\varphi_r:q_n,q_{n'}:\omega)=$$
$$\frac{2}{\pi i}\sqrt{\frac{1}{q_nq_{n'}r_sr_r}}\exp[i(q_nr_s+q_{n'}r_r)]\tilde{\psi}_{Scattered}(z_s,z_r,\varphi_s,\varphi_r:q_n,q_{n'}:\omega) \tag{28.30}$$

$$\widetilde{\psi}_{Scattered}(z_s, z_r, \varphi_s, \varphi_r : q_n, q_{n'} : \omega) =$$
$$\sum_N \sum_{s=+}^{-} \sum_{s'=+}^{-} \sum_{\sigma ml} \sum_{\sigma' m' l'} d_{Src,\sigma ml} d_{Rcv,\sigma' m' l'} \frac{\{\exp[(+ish_N + is'h'_N)(z_{N+1} - z_N)] - 1\}}{(+ish_N + is'h'_N)}$$
$$\frac{\rho(z_r)}{\rho_N} S_{Volume,N} Area \widetilde{A}^{s}_{\sigma ml,N_{Src},N}(z_s, \varphi_s : q_n) \widetilde{A}^{s'}_{\sigma' m' l',N_{Rcv},N}(z_r, \varphi_r : q_{n'}) \tag{28.31}$$

$$\widetilde{A}^{\pm}{}_{\sigma ml,N_{src},N}(z_s, \varphi_s : q_n) = \frac{2\pi i}{\rho(z_s)}(-i)^m \sqrt{\frac{\varepsilon(m)}{8\pi}} \left\{ \begin{matrix} \cos(m\varphi_s), \sigma = e \\ \sin(m\varphi_s), \sigma = o \end{matrix} \right\}$$
$$\{A^+_{N_{src}} F^+_{N_{src}}(z_s : q_n) B^+_{ml,N_{src}} + A^-_{N_{src}} F^-_{N_{src}}(z_s : q_n) B^-_{ml,N_{src}}\} A^{\pm}{}_N(q_n) \tag{28.32}$$

Substituting Equation 28.30 into Equation 28.29 and expanding the argument about the center frequency, keeping terms of quadratic order in the expansion, one arrives at the following approximation of the scattered field:

$$\psi_{Scattered}(r_s, r_r, z_s, z_r, \varphi_s, \varphi_r : q_n, q_{n'} : t) =$$
$$\frac{2}{\pi i} \sqrt{\frac{1}{q_{n0} q_{n0'} r_{sb} r_{rb}}} \widetilde{\psi}_{Scattered}(z_s, z_r, \varphi_s, \varphi_r : q_{n0}, q_{n'0} : \omega_0) \tag{28.33}$$
$$\exp[i(q_{n0} r_{sb} + q_{n'0} r_{rb}) - i\omega_0 t] \int_{-\infty}^{+\infty} d\omega \exp[-\alpha^2(\omega - \omega_0)^2 - i\beta(\omega - \omega_0)]$$

$$\alpha^2 = \frac{1}{2\Delta\omega^2} - i(q''_{n0} r_{sb} + q''_{n'0} r_{rb}) \tag{28.34}$$

$$\beta = t - (q'_{n0} r_{sb} + q'_{n'0} r_{rb}) \tag{28.35}$$

Upon completing the Gaussian in the integrand, one obtains the following asymptotic representation of the scattered field:

$$\psi_{Scattered}(r_s, r_r,, z_s, z_r, \varphi_s, \varphi_r : q_n, q_{n'} : t) =$$
$$\frac{2}{\pi i} \sqrt{\frac{1}{q_{n0} q_{n0'} r_s r_r}} \widetilde{\psi}_{Scattered}(z_s, z_r, \varphi_s, \varphi_r : q_{n0}, q_{n'0} : \omega_0) \tag{28.36}$$
$$\exp[i(q_{n0} r_s + q_{n'0} r_r) - i\omega_0 t_0] \frac{\sqrt{2\pi}}{\tau} \exp[-(t - t_0)^2 / 2\tau^2 - i\omega_0(t - t_0)]$$

$$t_0 = (q'_{n0} r_s + q'_{n'0} r_r) \tag{28.37}$$

$$\tau^2 = 2\alpha^2 = \frac{1}{\Delta\omega^2} - 2i(q''_{n0} r_s + q''_{n'0} r_r) \tag{28.38}$$

The parameters t_0 and τ are the effective arrival time and pulse-length of the scattered signal.

29. PLANE WAVE APPROXIMATION OF SIGNAL-TO-NOISE RATIO CALCULATIONS

This section uses the plane wave approximation of the normal modes to calculate signal-to-noise ratio (SNR). This approximation decomposes the normal mode contribution to the Greens' Function at the source and field point into upward and downward-going plane waves with a grazing angle given by the following relationship between vertical wavenumber and acoustic wavenumber at the source and field points:

$$\mathrm{Re}\left(\frac{h_{Src}(q)}{k_{Src}}\right) = \sin(\vartheta_{Src}) \tag{29.1}$$

The parameter ϑ_{Src} denotes the grazing angle of the normal mode at the source. A similar expression is used for the grazing angle of the normal mode at the field point. The vertical wavenumber is positive for downward going plane waves, and negative for upward going plane waves. Similarly, the grazing angle is positive for downward-going plane waves and negative for upward-going plane waves. This sign convention is related to the fact that the z-axis is directed downwards in PC SWAT.

One begins with the time domain representation of a normal mode. Propagation of a pulse is described by the Fourier Transform:

$$G_n(r,z_s,z:t) = \int_{-\infty}^{+\infty} S(\omega)G_n(r,z_s,z:\omega)e^{-i\omega t}d\omega \tag{29.2}$$

Here $S(\omega)$ is the spectrum of the incident signal. In the following instance, one may assume a spectrum of the following form:

$$S(\omega) = \exp\left(-\frac{(\omega-\omega_0)^2}{2\Delta\omega^2}\right) \tag{29.3}$$

Here, ω_0 is the center angular frequency, and $\Delta\omega$ is the angular bandwidth of the pulse. Propagation of an arbitrary band-limited pulse can be obtained by convolving the scattered signal from a broadband Gaussian pulse with the incident signal. The incident signal is of the following form:

$$s(t) = \int_{-\infty}^{+\infty} S(\omega)e^{-i\omega t}d\omega = \sqrt{2\pi}\Delta\omega\exp(-t^2\Delta\omega^2/2 - i\omega_0 t) \tag{29.4}$$

The parameter τ represents the pulse length of the incident signal.

$$\tau = \frac{1}{\Delta\omega} \tag{29.5}$$

Equation 15.17 describes the time domain representation of the Greens' Function, as shown below:

$$G_n(r,z_s,z:t) = \frac{i}{4\rho(z_s)} \sum_n \sqrt{\frac{2}{q_n r\pi}} \frac{\sqrt{\pi}}{\alpha} F(z_s:q_n)F(z:q_n)\exp(i(q_n r - \omega_0 t - \pi/4) - (t-t_n)^2/4\alpha^2) \tag{29.6}$$

Here, the reciprocal of the group velocity of the mode is:

$$q'_n(\omega) = \frac{dq_n(\omega)}{d\omega} \tag{29.7}$$

the rate of change of the reciprocal of the group velocity is:

$$q''_n(\omega) = \frac{d^2 q_n(\omega)}{d\omega^2} \tag{29.8}$$

the travel time for the mode is:

$$t_n = rq'_n(\omega_0) \tag{29.9}$$

and, the complex width squared of the effective Gaussian is:

$$\alpha^2 = \frac{1}{2\Delta\omega^2} - \frac{i}{2}q''_n(\omega_0) \tag{29.10}$$

The normal mode terms in Equation 29.6 can be decomposed into an upward and downward-going plane wave:

$$G_n^{\pm,\pm}(r,z_s,z:t) =$$
$$\frac{i}{4\rho(z_s)} \sqrt{\frac{2}{q_n r\pi}} \frac{\sqrt{\pi}}{\alpha} F^{\pm}(z_s:q_n)F^{\pm}(z:q_n)\exp(i(q_n r - \omega_0 t - \pi/4) - (t-t_n)^2/4\alpha^2) \tag{29.11}$$

Here, the function $F^{\pm}(z:q)$ represents the decomposition of the depth function into upward and downward-going plane waves, as defined below in terms of the components of the propagator matrix in the N'th layer containing the depth z:

$$F^{\pm}(z:q) = A_N^{\pm}(q)F_N^{\pm}(z:q) = A_N^{\pm}(q)\exp[\pm ih_N(q)(z-z_N)] \tag{29.12}$$

The normal mode contribution to the signal from the target is of the following form:

$$\psi^{s,s',s'',s'''}{}_{Scattered}(r_s, r_r, z_s, z_r, z, \varphi_s, \varphi_r : q_n, q_{n'} : t+t') = (4\pi)^2 A_{Src} D_{Src}(s\vartheta_{Src}, \pi - \varphi_s)$$
$$D_{Rcv}(s'''\vartheta_{Rcv}, \pi - \varphi_r) G_n^{s,s'}(r_s, z_s, z : t) G_n^{s'',s'''}(r_r, z, z_r : t') S_{Target}(s'\vartheta_n, \pi - \varphi_s, s''\vartheta_{n'}, \varphi_r) \quad (29.13)$$

The parameter $A_{Src} = 10^{SL/20}$ is the amplitude of the incident field at a unit distance from the source, where *SL* is the source level of the projector in decibels. The vectors (r_s, z_s, φ_s) and s (r_r, z_r, φ_r) are the cylindrical coordinates of the source and receiver relative to the target, that is, r_s is the range from the target to the source, z_s is the depth of the source, and φ_s is the bearing of the source as seen from the target. Similarly, r_r is the range from the target to the receiver, z_r is the depth of the receiver, and φ_r is the bearing of the receiver as seen from the target. The functions $G_n^{s,s'}(r, z, z' : t)$ are the decomposition of the n'th normal mode contribution to the Greens' Function in terms of upward and downward going plane waves at the two endpoints. The angle:

$$\vartheta_n = \sin^{-1}(\mathrm{Re}(\frac{h(q_n)}{k})) \quad (29.14)$$

is the grazing angle of the n'th normal mode at the target. The angles ϑ_{Src} and ϑ_{Rcv} are the grazing angles for the outgoing and incoming normal mode at the source and receiver, respectively. The function $S_{Target}(\vartheta, \varphi, \vartheta', \varphi')$ is the bistatic scattering function for the target. In the case of an omni-directional point target, this function is equal to the constant $10^{TS/20}$, where *TS* is the target strength in decibels.

In general, one estimates the signal from the target by incoherently adding the magnitude squared of all the normal mode contributions given by Equation 29.13 that arrive at the receiver within a pulse length of each other.

The next step in evaluating the signal-to-noise ratio of a system is to estimate the amount of surface, bottom, and volume reverberation arriving at the receiver as a function of travel time.

First, consider the case of calculating the mean square surface reverberation. The magnitude squared of the normal mode contribution to surface reverberation is of the following form:

$$|\psi^{s,s'}{}_{Surface}(r_s, r_r, z_s, z_r, z=0, \varphi : q_n, q_{n'} : t+t')|^2 =$$
$$(4\pi)^4 |A_{Src}|^2 |D_{Src}(s\vartheta_{Src}, 0) D_{Rcv}(s'\vartheta_{Rcv}, \varphi)|^2 |G_n^{s,-}(r_s, z_s, z=0 : t) G_{n'}^{+,s'}(r_r, z=0, z_r : t')|^2 \quad (29.14)$$
$$S_{Surface}(\vartheta_n, \vartheta_{n'}, \pi - \varphi)(v_{Src} r_s \frac{dq_n}{d\omega} \Delta\vartheta_{Hor} \Delta R)$$

Here, the quantity $S_{Surface}(\vartheta, \vartheta', \varphi)$ is the bistatic surface scattering strength.

The term:

$$Area = v_{Src} r_s \frac{dq_n}{d\omega} \Delta\vartheta_{Hor} \Delta R \tag{29.15}$$

is the area ensonified by the incident signal, where ΔR is the range resolution of the signal, and $\Delta\vartheta_{Hor}$ is the horizontal beam-width of the system. The parameter r_s is the range from the projector to the ensonified area on the surface, and z_s is the depth of the source. Similarly, r_r is the range from the receiver to the ensonified area, and z_r is the depth of the receiver. The parameter z is the depth of the surface. The angle φ is the azimuthal angle between the projector and the receiver, as seen from the center of the ensonified area. For a monostatic, active sonar, this angle is approximately zero.

In the discussion of bottom reverberation, one may decompose the bottom reverberation into two parts. The first describes bottom reverberation due to the roughness of the interface. The second describes bottom reverberation due to volume inhomogeneities in the sediment. High frequency models of bottom reverberation usually combine these two contributions into a single scattering function, since the sound is generally limited to a narrow layer about the interface owing to the attenuation of sound in the sediment. In the case of low frequency propagation, the sound has significant penetration into the sediment, and scattering from volume inhomogeneities cannot be considered to arise from scattering from a thin layer about the interface.

$$\begin{aligned} &|\psi^{s,s'}{}_{Bottom}(r_s, r_r, z_s, z_r, z_b, \varphi : q_n, q_{n'} : t+t')|^2 = \\ &(4\pi)^4 \, |A_{Src}|^2 |D_{Src}(s\vartheta_{Src}, 0) D_{Rcv}(s'\vartheta_{Rcv}, \varphi)|^2 |G_n^{s,+}(r_s, z_s, z_b : t) G_{n'}^{-,s'}(r_r, z_b, z_r : t')|^2 \\ &S_{Bottom}(\vartheta_n, \vartheta_{n'}, \varphi_r - \varphi_s + \pi)(v_{Src} r_s \frac{dq_n}{d\omega} \Delta\vartheta_{Hor} \Delta R) \end{aligned} \tag{29.16}$$

Here, the quantity $S_{Bottom}(\vartheta, \vartheta', \varphi)$ is the bistatic bottom scattering strength for the rough interface. The term:

$$Area = v_{Src} r_s \frac{dq_n}{d\omega} \Delta\vartheta_{Hor} \Delta R \tag{29.17}$$

is the area ensonified by the incident signal, where ΔR is the range resolution of the signal, and $\Delta\vartheta_{Hor}$ is the horizontal beam-width of the system. The parameter r_s is the range from the projector to the ensonified area, and z_s is the depth of the source. Similarly, r_r is the range from the receiver to the ensonified area, and z_r is the depth of the receiver. The angle φ is the azimuthal angle between the projector and the receiver, as seen from the center of the ensonified area. The parameter z is the depth of the bottom interface. In the case of a multi-layered bottom, one can easily sum over the scattering from each interface.

The magnitude squared of the mean normal mode contribution due to the scattering from volume inhomogeneities in the N'th layer is of the following form:

$$
\begin{aligned}
&|\psi^{s,s',s'',s'''}{}_{Volume}(r_s,r_r,z_s,z_r,z,\varphi:q_n,q_{n'}:t+t')|^2 = \\
&(4\pi)^4\,|A_{Src}|^2|D_{Src}(s\vartheta_{Src},0)D_{Rcv}(s'''\vartheta_{Rcv},\varphi)|^2|\rho(z_r)/\rho_N|^2|\widetilde{G}_n^{s,s'}(r_s,z_s:t)\widetilde{G}_n^{s''',s''}(r_r,z_r:t')|^2 \\
&(v_{Src}r_s\frac{dq_n}{d\omega}\Delta\vartheta_{Hor}\Delta R)S_{Volume}\left\|\left[\frac{\exp[\{+is'h_N(q_n)+is'h'_N(q_{n'})\}(z_{N+1}-z_N)]-1}{(+is''h_N(q_n)+is''h'_N(q_{n'}))}\right]\right\|^2
\end{aligned}
\tag{29.18}
$$

The function $\widetilde{G}_n^{\pm,\pm}(r,z_s:t)$ is the following function, where the coefficients $A_N^{\pm}(q)$ are the coefficients of the depth function in the N'th layer:

$$
\begin{aligned}
&\widetilde{G}_n^{s,s'}(r,z_s:t) = \\
&\frac{i}{4\rho(z_s)}\sqrt{\frac{2}{q_n r\pi}}\frac{\sqrt{\pi}}{\alpha}F^s(z_s:q_n)A_N^{s'}(q_n)\exp(i(q_n r-\omega_0 t-\pi/4)-(t-t_n)^2/4\alpha^2)
\end{aligned}
\tag{29.19}
$$

These functions arise by integrating the product of the two Greens' Functions over the depth of the scatterer:

$$
\begin{aligned}
&\int_{z_N}^{z_{N+1}}dzG_n^{s,s'}(r_s,z_s,z:t)G_{n'}^{s'',s'''}(r_r,z,z_r:t') = \\
&\frac{\rho(z_r)}{\rho_N}\int_{z_N}^{z_{N+1}}dzG_n^{s,s'}(r_s,z_s,z:t)G_{n'}^{s''',s''}(r_r,z_r,z:t') \\
&\frac{\rho(z_r)}{\rho_N}\widetilde{G}_n^{s,s'}(r_s,z_s:t)\widetilde{G}_{n'}^{s''',s''}(r_r,z_r:t')\left[\frac{\exp[\{+is'h_N(q_n)+is''h'_N(q_{n'})\}(z_{N+1}-z_N)]-1}{(+is'h_N(q_n)+is''h'_N(q_{n'}))}\right]
\end{aligned}
\tag{29.20}
$$

One may use the following expression for the upward and downward-going components of the depth function in terms of exponentials and the coefficients of the propagator matrix in evaluating the above integral.

$$
F^{\pm}(z:q) = A_N^{\pm}(q)\exp[\pm ih_N(q)(z-z_N)]
\tag{29.21}
$$

An alternative expression for the scattering from volume inhomogeneities is to replace Equation 29.18 by the following expression:

$$|\psi^{s,s',s'',s'''}{}_{Volume}(r_s,r_r,z_s,z_r,z,\varphi:q_n,q_{n'}:t+t')|^2 =$$
$$(4\pi)^4|A_{Src}|^2|D_{Src}(s\vartheta_{Src},0)D_{Rcv}(s'''\vartheta_{Rcv},\varphi)|^2|\rho(z_r)/\rho_N|^2|\widetilde{G}_n^{s,s'}(r_s,z_s:t)\widetilde{G}_{n'}^{s''',s''}(r_r,z_r:t')|^2$$
$$(v_{Src}r_s\frac{dq_n}{d\omega}\Delta\vartheta_{Hor}\Delta R(z_{N+1}-z_N))S_{Volume}\frac{(1-\exp[-(2s'\mathrm{Im}(h)+2s''\mathrm{Im}(h'))(z_{N+1}-z_N)])}{(2s'\mathrm{Im}(h)+2s''\mathrm{Im}(h'))(z_{N+1}-z_N)}$$

(29.22)

In deriving this expression one replaces Equation 29.20 by the following incoherent sum of the magnitude squared of the Greens' Function over depth, that is, one performs a summation over the energy rather than pressure as a function of depth:

$$\int_{z_N}^{z_{N+1}}dz\left|G_n^{s,s'}(r_s,z_s,z:t)G_{n'}^{s'',s'''}(r_r,z,z_r:t')\right|^2 = \int_{z_N}^{z_{N+1}}dz\left|\frac{\rho(z_r)}{\rho_N}G_n^{s,s'}(r_s,z_s,z't)G_{n'}^{s''',s''}(r_r,z_r,z:t')\right|^2$$
$$\left|\frac{\rho(z_r)}{\rho_N}\widetilde{G}_n^{s,s'}(r_s,z_s:t)\widetilde{G}_{n'}^{s''',s''}(r_r,z_r:t')\right|^2\int_{z_N}^{z_{N+1}}dz\exp[-(2s'\mathrm{Im}(h)+2s''\mathrm{Im}(h'))(z-z_N)] =$$
$$\left|\frac{\rho(z_r)}{\rho_N}\widetilde{G}_n^{s,s'}(r_s,z_s:t)\widetilde{G}_{n'}^{s''',s''}(r_r,z_r:t')\right|^2(z_{N+1}-z_N)\frac{(1-\exp[-(2s'\mathrm{Im}(h)+2s''\mathrm{Im}(h'))(z_{N+1}-z_N)])}{(2s'\mathrm{Im}(h)+2s''\mathrm{Im}(h'))(z_{N+1}-z_N)}$$

(29.23)

In the limit the imaginary component of the vertical wavenumber vanishes the following limit is obtained:

$$\frac{(1-\exp[-(2s'\mathrm{Im}(h)+2s''\mathrm{Im}(h'))(z_{N+1}-z_N)])}{(2s'\mathrm{Im}(h)+2s''\mathrm{Im}(h'))(z_{N+1}-z_N)}\rightarrow 1 \qquad (29.24)$$

The quantity S_{Volume} is the volume scattering strength for volume inhomogeneities in the N'th layer. The term:

$$Volume = v_{Src}r_s\frac{dq_n}{d\omega}\Delta\vartheta_{Hor}\Delta R(z_{N+1}-z_N) \qquad (29.25)$$

is the volume ensonified by the incident signal. The parameter r_s is the range from the projector to the ensonified volume, and z_s is the depth of the source. Similarly, r_r is the range from the receiver to the ensonified volume, and z_r is the depth of the receiver. The angle φ is the azimuthal angle between the projector and the receiver, as seen from the center of the ensonified volume. In the case of a multi-layered bottom, one can easily sum over the scattering of the volume inhomogeneities in each layer. Scattering from volume inhomogeneities in the water

column (volume reverberation) is similarly treated. One must implicitly assume the directivity of the beam is approximately constant over the depth of the layer, thus application of Equation 29.18 or 29.22 may require sub-dividing the layers into smaller layers.

Calculation of the signal-to-noise ratio begins with calculating the return from the target at a fixed range by adding those returns from the target that return within a pulse length of each other using Equation 29.13. This process produces the envelope of the time series representing the different multipath returns from the target. Next one calculates the scattering from the surface and the bottom by integrating Equations 29.14 and 29.16 with respect to the range from the projector to the ensonified area on the surface and bottom respectively. Again, one adds contributions to the envelope of the surface and bottom reverberation that lie within a pulse length of each other. This process produces the envelope of a time series representing the scattering from the surface and the bottom. Next, one uses either Equation 29.18 or 29.22 to compute the scattering from volume inhomogeneities in the sediment by integrating these equations with respect to the range from the projector to the ensonified volume in the sediment. Similarly, one integrates either Equation 29.18 or 29.22 with respect to range from the projector to the ensonified volume to obtain the scattering from volume inhomogeneities in the water column. Finally, one adds the ambient noise term to the sum of the surface, bottom, and volume reverberation levels to determine the total noise in the system. The signal-to-noise ratio for the system can be estimated by taking the ratio of the largest return from the target and the total noise arriving at the sonar at the same time.

30. COMPUTATION OF NORMAL MODES

Direct application of the characteristic equations in Sections 8 and 9 lead to numerical instability in the case of large sound speed gradients in the water column. This section describes a numerically stable technique for solving the characteristic equation for the normal modes in a waveguide.

Suppose one has a waveguide consisting of N homogeneous layers, where the following matrices transform the depth coefficient at a given interface into the pressure and normal displacement at that interface:

$$M_{1,1}^{N+1,N} = \exp(+ih_N d_N) \tag{30.1a}$$

$$M_{1,2}^{N+1,N} = \exp(-ih_N d_N) \tag{30.1b}$$

$$M_{2,1}^{N+1,N} = +I\frac{h_N}{\rho_N}\exp(+ih_N d_N) \tag{30.1c}$$

$$M_{2,2}^{N+1,N} = -I\frac{h_N}{\rho_N}\exp(-ih_N d_N) \tag{30.1d}$$

$$M_{1,1}^{N,N} = +1 \tag{30.1e}$$

$$M_{1,2}^{N,N} = +1 \tag{30.1f}$$

$$M_{2,1}^{N,N} = +I\frac{h_N}{\rho_N} \tag{30.1g}$$

$$M_{2,2}^{N,N} = -I\frac{h_N}{\rho_N} \tag{30.1h}$$

Suppose the depth function in the N'th layer is of the following form:

$$F_N(z) = A_N^+ \exp(+ih_N(z-z_N)) + A_N^- \exp(-ih_N(z-z_N)) \tag{30.2}$$

The equations of continuity of pressure and the normal displacement at the N'th interface are of the following form:

$$0 = \begin{pmatrix} M_{1,1}^{N,N-1} & M_{1,2}^{N,N-1} \\ M_{2,1}^{N,N-1} & M_{2,2}^{N,N-1} \end{pmatrix} \begin{pmatrix} A_{-N-1}^{+} \\ A_{N-1}^{-} \end{pmatrix} - \begin{pmatrix} M_{1,1}^{N,N} & M_{1,2}^{N,N} \\ M_{2,1}^{N,N} & M_{2,2}^{N,N} \end{pmatrix} \begin{pmatrix} A_{-N}^{+} \\ A_{N}^{-} \end{pmatrix} \tag{30.3}$$

In the case of a rigid bottom, one can express the equations of continuity and the boundary condition at the pressure release surface and the rigid bottom by the following set of linear equations:

$$0 = CA \tag{30.4}$$

$$A^T = \left(A_o^+, A_o^-, A_1^+, A_1^-, \ldots A_N^+, A_N^-\right) \tag{30.5}$$

The global matrix C is the matrix whose first row is given by the following expression that encapsulates the boundary condition of a pressure release surface.

$$C_{1,1\ldots 2N} = (1,1,0,0,\ldots,0,0) \tag{30.6}$$

The last row of the global matrix C takes on the following for and encapsulates the boundary condition of a rigid bottom:

$$C_{2N,1\ldots 2N} = \left(0,0,\ldots,+i\frac{h_N}{\rho_N}\exp(+ih_N d_N),-i\frac{h_N}{\rho_N}\exp(-ih_N d_N)\right) \tag{30.7}$$

The intermediary rows of the global matrix C express the equations of continuity of pressure and normal displacement in the intermediary interfaces. The non-zero elements of the global matrix C in the intermediary rows are of the following form:

$$C_{2n+i+1,2n+j+1} = +M_{i,j}^{n,n} \tag{30.8a}$$

$$C_{2(n+1)+i+1,2n+j+1} = -M_{i,j}^{n+1,n} \tag{30.8b}$$

The global matrix C is extremely sparse and has the following quasi-diagonal form.

$$C = \begin{pmatrix} 1 & 1 & 0 & 0 & 0 & 0 & \\ M_{1,1}^{1,0} & M_{1,2}^{1,0} & -M_{1,1}^{1,1} & -M_{1,2}^{1,1} & 0 & 0 & \\ M_{2,1}^{1,0} & M_{2,2}^{1,0} & -M_{2,1}^{1,1} & -M_{2,2}^{1,1} & 0 & 0 & \cdots \\ 0 & 0 & M_{1,1}^{2,1} & M_{1,2}^{2,1} & -M_{1,1}^{2,2} & -M_{1,2}^{2,2} & \\ 0 & 0 & M_{2,1}^{2,1} & M_{2,2}^{2,1} & -M_{2,1}^{2,2} & -M_{2,2}^{2,2} & \end{pmatrix} \tag{30.9}$$

In the case the bottom layer is a homogeneous half space the last row in the global matrix is of the following form.

$$C_{2N,1\ldots 2N} = (0,0,\ldots,0,1) \tag{30.10}$$

This equation encapsulates the boundary condition that the upward-going component of the depth function in the homogeneous half space vanishes.

Note, terms with a negative exponential in the global matrix C may be eliminated by rescaling the columns of the matrix. In this manner, the off-diagonal terms are either Order 1 or they are exponentially damped. The characteristic equation for the normal modes is found by tri-diagonalizing the global matrix C and taking the product of the remaining diagonal terms to obtain the determinant of the global matrix C. The normal modes are those values of the horizontal wavenumber for which this determinant vanishes.

PC SWAT 7.0 solves this characteristic equation for the normal modes in the case the sound speed is real by using a combination of the method of bisection and Newton's Method to obtain the solution for the normal modes in the absence of attenuation. PC SWAT then uses perturbation theory of the normal modes to estimate the complex eigenvalues due to attenuation in the water column. The resulting normal modes have a small imaginary component corresponding to the attenuation of the mode. Attenuation of the normal mode in the sediment is handled by altering the vertical wavenumber in the sediment, which generally has a significant positive imaginary component. The effects of attenuation in the sediment on the horizontal wavenumber are ignored, since the attenuation is generally large enough that perturbation theory is not applicable.

In the case of a rigid bottom, the coefficients of the depth functions are found by solving a linear equation of the following form:

$$V = GB \tag{30.11}$$

$$G = \begin{pmatrix} -M_{1,1}^{1,1} & -M_{1,2}^{1,1} & 0 & 0 & 0 & 0 & \\ -M_{2,1}^{1,1} & -M_{2,2}^{1,1} & 0 & 0 & 0 & 0 & \\ M_{1,1}^{2,1} & M_{1,2}^{2,1} & -M_{1,1}^{2,2} & -M_{1,2}^{2,2} & 0 & 0 & \cdots \\ M_{2,1}^{2,1} & M_{2,2}^{2,1} & -M_{2,1}^{2,2} & M_{2,2}^{2,2} & 0 & 0 & \\ 0 & 0 & M_{1,1}^{3,2} & M_{1,2}^{3,2} & -M_{1,1}^{3,3} & -M_{1,2}^{3,3} & \end{pmatrix} \tag{30.12}$$

$$B = (A_1^+, A_1^-, \ldots, A_N^+, A_N^-) \tag{30.13}$$

$$V_1 = -(M^{1,0}\begin{pmatrix} +1 \\ -1 \end{pmatrix})_1 \tag{30.14a}$$

$$V_2 = -(M^{1,0}\begin{pmatrix} +1 \\ -1 \end{pmatrix})_2 \tag{30.14b}$$

$$V_i = 0, i \neq 1,2 \tag{30.14c}$$

The matrix G is the 2(N-1) x 2(N-1) dimensional sub-matrix of the global matrix C obtained by eliminating the first two columns and the first and last rows. The vector V represents the source term due to the depth coefficients in the top layer, where one must assume the depth coefficients in the top layer are given by the following expression:

$$\begin{pmatrix} A_0^+ \\ A_0^- \end{pmatrix} = \begin{pmatrix} +1 \\ -1 \end{pmatrix} \tag{30.15}$$

PC SWAT solves the above equations by using Gaussian elimination with partial pivoting to find the inverse of the matrix G. The coefficients of the depth function in layers 1,2,...N are then obtained by the following matrix multiplication:

$$(A_1^+, A_1^-, ..., A_N^+, A_N^-)^T = G^{-1}V \tag{30.16}$$

In the case of a homogeneous half space, PC SWAT forms the matrix 2(N-2) x 2(N-2) dimensional sub-matrix of the matrix C by eliminating the first two columns and the last two columns, the first row, and the last three rows. The vector B is the 2(N-2) vector containing the depth coefficients for layers 1,2,...(N-1). The depth coefficient in the homogeneous layer is obtained by using the propagator matrices to relate the depth coefficients in the (N-1)'th layer to the N'th layer.

31. SAMPLE CALCULATIONS

This section describes a set of test cases of the low frequency propagation model. This model is described in Reference 1.

Table 31-1 displays a comparison of the complex Green's Function for a rigid waveguide of depth 300 m calculated by PC SWAT and the exact normal mode solution at a frequency of 200 Hz. The source is located at mid water column. The range to the field point is 300 m. As can be seen from Table 31-1, PC SWAT agrees with the exact solution to six significant figures.

TABLE 31-1. COMPARISON OF EXACT AND APPROXIMATE SOLUTION OF THE GREEN'S FUNCTION

	PCSWAT		EXACT	
Depth, M	**Real**	**Imaginary**	**Real**	**Imaginary**
0	2.156952E-21	-1.742372E-48	0.000000E+00	0.000000E+00
3.1	-7.601606E-06	-7.848293E-05	-7.601606E-06	-7.848293E-05
6.2	-9.370370E-04	1.848949E-04	-9.370370E-04	1.848949E-04
9.3	5.937188E-04	5.420785E-05	5.937188E-04	5.420785E-05
12.4	4.005531E-04	-2.861161E-04	4.005531E-04	-2.861161E-04
15.5	-3.460245E-05	3.103788E-04	-3.460245E-05	3.103788E-04
18.6	7.772559E-05	-1.685571E-04	7.772559E-05	-1.685571E-04
21.7	-7.101162E-04	-2.997450E-04	-7.101162E-04	-2.997450E-04
24.8	2.578272E-05	3.199933E-04	2.578272E-05	3.199933E-04
27.9	3.287343E-04	-5.274567E-05	3.287343E-04	-5.274567E-05
31	2.759619E-04	3.640188E-04	2.759619E-04	3.640188E-04
34.1	-3.689313E-06	6.981516E-06	-3.689313E-06	6.981516E-06
37.2	-2.695758E-04	-5.676645E-04	-2.695758E-04	-5.676645E-04
40.3	1.927261E-04	-1.785782E-04	1.927261E-04	-1.785782E-04
43.4	-3.423437E-04	8.770414E-05	-3.423437E-04	8.770414E-05
46.5	-6.500707E-05	5.725958E-04	-6.500707E-05	5.725958E-04
49.6	8.186082E-05	2.823280E-05	8.186082E-05	2.823280E-05
52.7	-1.378475E-04	-9.820521E-05	-1.378475E-04	-9.820521E-05
55.8	7.905617E-04	2.852105E-05	7.905617E-04	2.852105E-05
58.9	7.423926E-05	-2.768488E-04	7.423926E-05	-2.768488E-04
62	-6.218280E-04	2.453624E-05	-6.218280E-04	2.453624E-05
65.1	8.851132E-05	-4.144703E-04	8.851132E-05	-4.144703E-04
68.2	-1.450531E-04	-2.095526E-05	-1.450531E-04	-2.095526E-05
71.3	-3.129984E-04	4.758295E-04	3.129984E-04	4.758295E-04
74.4	1.079975E-04	2.570571E-04	1.079975E-04	2.570571E-04
77.5	-2.536621E-05	1.390918E-04	-2.536621E-05	1.390918E-04
80.6	2.644454E-04	-2.384132E-04	2.644454E-04	-2.384132E-04

TABLE 31-1. COMPARISON OF EXACT AND APPROXIMATE SOLUTION OF THE GREEN'S FUNCTION, CONTINUED

	PCSWAT		EXACT	
Depth, M	Real	Imaginary	Real	Imaginary
83.7	5.999015E-04	4.540631E-04	5.999015E-04	4.540631E-04
86.8	-1.663070E-04	-2.418384E-04	-1.663070E-04	-2.418384E-04
89.9	-3.274314E-05	-7.096214E-04	-3.274314E-05	-7.096214E-04
93	3.078189E-04	2.401429E-04	3.078189E-04	2.401429E-04
96.1	-2.750968E-04	-4.493027E-04	-2.750968E-04	-4.493027E-04
99.2	-3.570023E-04	-2.424585E-05	-3.570023E-04	-2.424585E-05
102.3	-3.382565E-04	5.974809E-05	-3.382565E-04	5.974809E-05
105.4	2.291967E-05	-2.152185E-04	2.291967E-05	-2.152185E-04
108.5	-1.661176E-04	5.873839E-04	-1.661176E-04	5.873839E-04
111.6	-5.834619E-04	4.815931E-05	-5.834619E-04	4.815931E-05
114.7	3.460832E-04	2.040280E-04	3.460832E-04	2.040280E-04
117.8	1.763182E-04	2.856471E-04	1.763182E-04	2.856471E-04
120.9	-3.589055E-04	2.928459E-04	-3.589055E-04	2.928459E-04
124	3.807956E-04	3.962045E-04	3.807956E-04	3.962045E-04
127.1	3.738250E-04	-3.303400E-04	3.738250E-04	-3.303400E-04
130.2	-3.827622E-05	5.404839E-04	-3.827622E-05	5.404839E-04
133.3	3.019754E-04	2.317492E-04	3.019754E-04	2.317492E-04
136.4	3.601126E-04	-4.246805E-04	3.601126E-04	-4.246805E-04
139.5	1.806243E-04	3.596155E-04	1.806243E-04	3.596155E-04
142.6	3.077695E-04	3.466247E-05	3.077695E-04	3.466247E-05
145.7	1.474091E-04	5.509475E-07	1.474091E-04	5.509475E-07
148.8	4.343452E-04	-2.044144E-04	4.343452E-04	-2.044144E-04
151.9	2.670685E-04	9.472733E-05	2.670685E-04	9.472733E-05
155	-6.337753E-05	3.846362E-04	-6.337753E-05	3.846362E-04
158.1	7.366325E-04	-5.794180E-04	7.366325E-04	-5.794180E-04
161.2	5.727772E-05	2.611416E-04	5.727772E-05	2.611416E-04
164.3	-1.872883E-05	4.375409E-04	-1.872883E-05	4.375409E-04
167.4	7.384581E-04	-3.635707E-04	7.384581E-04	-3.635707E-04
170.5	-7.500781E-05	2.628381E-04	-7.500781E-05	2.628381E-04
173.6	1.815121E-04	3.189365E-04	1.815121E-04	3.189365E-04
176.7	2.031955E-04	2.004866E-04	2.031955E-04	2.004866E-04
179.8	9.574840E-05	1.482104E-04	9.574840E-05	1.482104E-04
182.9	1.778534E-04	2.029185E-04	1.778534E-04	2.029185E-04
186	-5.309431E-04	5.078087E-04	-5.309431E-04	5.078087E-04
189.1	2.515085E-04	7.838585E-05	2.515085E-04	7.838585E-05
192.2	-2.746774E-05	6.725184E-05	-2.746774E-05	6.725184E-05
195.3	-8.236512E-04	2.094750E-04	-8.236512E-04	2.094750E-04
198.4	1.617321E-05	-7.181784E-05	1.617321E-05	-7.181784E-05
201.5	2.697329E-05	-5.519383E-05	2.697329E-05	-5.519383E-05
204.6	-3.961550E-04	-3.287576E-04	-3.961550E-04	-3.287576E-04
207.7	-2.016652E-04	-3.599796E-04	-2.016652E-04	-3.599796E-04
210.8	4.106582E-04	5.658139E-05	4.106582E-04	5.658139E-05

TABLE 31-1. COMPARISON OF EXACT AND APPROXIMATE SOLUTION OF THE GREEN'S FUNCTION, CONTINUED

	PCSWAT		EXACT	
Depth, M	Real	Imaginary	Real	Imaginary
213.9	1.827767E-04	-2.275882E-04	1.827767E-04	-2.275882E-04
217	9.763994E-05	-3.040782E-04	9.763994E-05	-3.040782E-04
220.1	3.429965E-04	2.782708E-04	3.429965E-04	2.782708E-04
223.2	-8.857475E-05	4.557982E-04	-8.857475E-05	4.557982E-04
226.3	3.194940E-04	1.128602E-04	3.194940E-04	1.128602E-04
229.4	-2.846170E-04	-1.218884E-04	-2.846170E-04	-1.218884E-04
232.5	-6.002076E-04	3.137906E-04	-6.002076E-04	3.137906E-04
235.6	1.232101E-04	3.605080E-05	1.232101E-04	3.605080E-05
238.7	-1.108531E-04	-6.004108E-04	-1.108531E-04	-6.004108E-04
241.8	2.773905E-04	-6.520506E-05	2.773905E-04	-6.520506E-05
244.9	-1.062994E-04	-1.729520E-04	-1.062994E-04	-1.729520E-04
248	1.828837E-04	1.295178E-04	1.828837E-04	1.295178E-04
251.1	6.021326E-04	6.909265E-04	6.021326E-04	6.909265E-04
254.2	-5.457961E-04	-3.855391E-04	-5.457961E-04	-3.855391E-04
257.3	-1.582388E-04	-1.412799E-05	-1.582388E-04	-1.412799E-05
260.4	-2.968068E-04	3.720943E-04	-2.968068E-04	3.720943E-04
263.5	-1.973936E-05	-5.479777E-04	-1.973936E-05	-5.479777E-04
266.6	6.874971E-04	-5.917463E-05	6.874971E-04	-5.917463E-05
269.7	-1.082978E-04	1.171910E-05	-1.082978E-04	1.171910E-05
272.8	-7.321751E-05	-1.322922E-06	-7.321751E-05	-1.322922E-06
275.9	-2.318374E-04	6.503385E-04	-2.318374E-04	6.503385E-04
279	5.030232E-05	-4.674538E-05	5.030232E-05	-4.674538E-05
282.1	4.462274E-05	-5.669501E-04	4.462274E-05	-5.669501E-04
285.2	-1.375995E-04	-1.844840E-04	-1.375995E-04	-1.844840E-04
288.3	3.717545E-04	9.333937E-05	3.717545E-04	9.333937E-05
291.4	-3.635994E-04	4.157779E-04	-3.635994E-04	4.157779E-04
294.5	2.495043E-04	2.992173E-04	2.495043E-04	2.992173E-04
297.6	9.235150E-05	-3.025371E-04	9.235150E-05	-3.025371E-04

Figure 31.1 presents a comparison of the coherent transmission loss between the high frequency and low frequency models in PC SWAT 7. The waveguide is 100 m deep with a rigid bottom and pressure release surface. The source and receiver are both located at a depth of 50 m. The frequency of the projector is 1000 Hz. Table 31-2 contains the sound velocity profile for the waveguide. In the case of the high frequency model a user-defined surface and reflection loss of 0 decibels is used.

TABLE 31-2. SOUND VELOCITY PROFILE OF TEST CASE

Depth (m)	Sound Speed (m/s)
0	1500
50	1460
100	1500

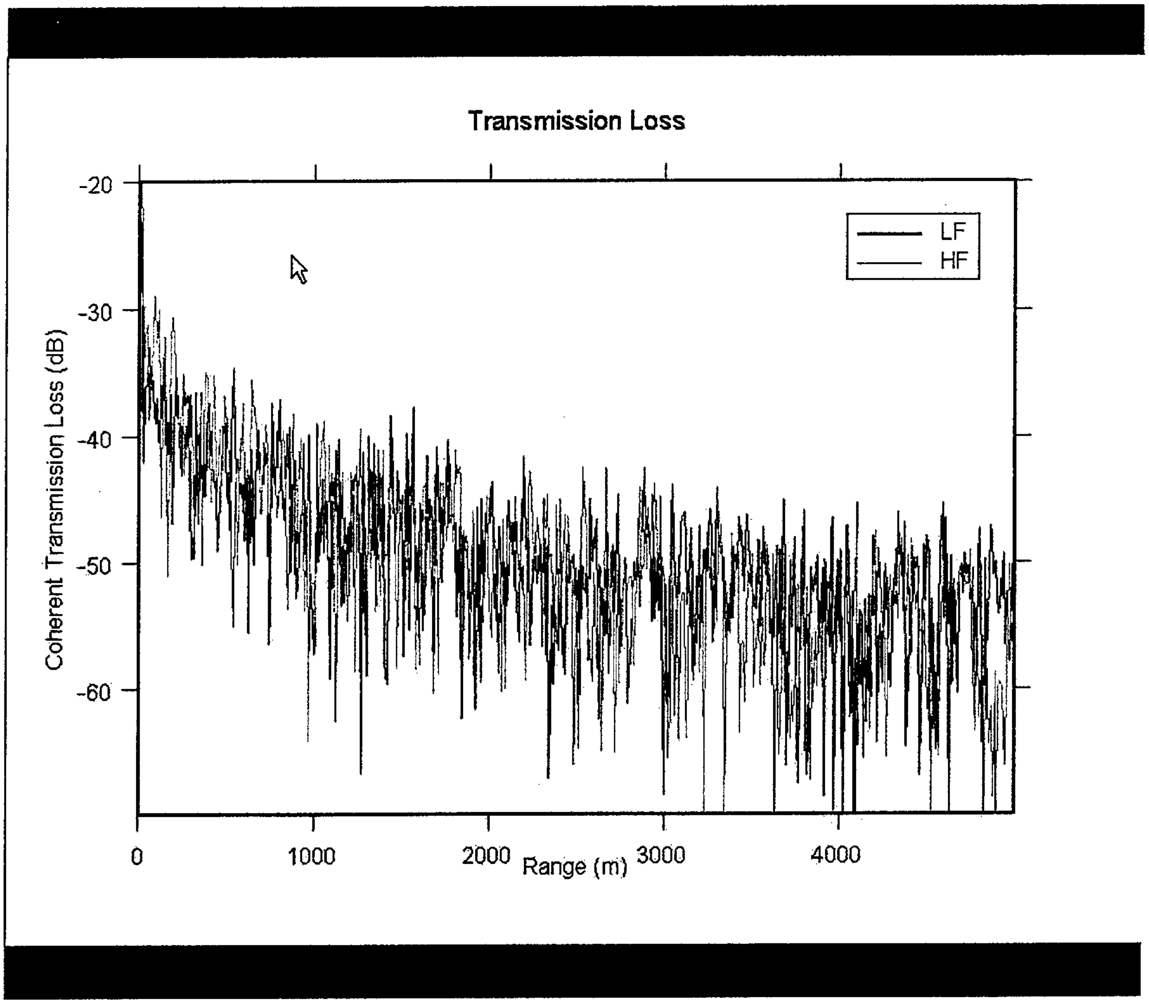

FIGURE 31.1 COMPARISON OF TRANSMISSION LOSS FOR HIGH FREQUENCY AND LOW FREQUENCY PROPAGATION MODELS

Note, the incoherent transmission loss for the high frequency and low frequency propagation models are not directly comparable, since each ray trajectory in the high frequency model represents a coherent sum of normal modes.

32. REFERENCES

1. Brekhovskikh, L.M., and O.A. Godin, *Acoustics of Layered Media I: Plane and Quasi-Plane Waves*, Springer Series on Wave Phenomena, Springer Verlag, New York, 1989.

2. Brekhovskikh, L.M., and O.A. Godin, *Acoustics of Layered Media I: Point Sources and Bounded Beams,* Springer Series on Wave Phenomena, Springer Verlag, New York, 1989.

3. Morse, Phillip M., and Herman Feshbach, *Methods of Theoretical Physics: Volume 1*, McGraw Hill Book Company, 1953, New York.

4. Morse, Phillip M., and Herman Feshbach, *Methods of Theoretical Physics: Volume 2*, McGraw Hill Book Company, 1953, New York.

5. Jensen, Finn B., William A. Kuperman, Michael B. Porter, and Henrik Schmidt, *Computational Ocean Acoustics: AI P Series in Modern Acoustics and Signal Processing*, AIP Press, New York, 1994.

6. Ewing, W.M., W.S. Jardetsky, and F. Press, *Elastic Waves in Layered Media*, McGraw-Hill Book Company, New York, 1957.

7. Stickler, D.C., "Normal-mode Program with Both the Discrete and Branch Line Contributions", Journal of the Acoustic Society of America. 57(4), pp 856-861.

8. Kuperman, W.A., "Coherent Component of Specular Reflection and Transmission at a Randomly Rough Two Fluid Interface", J. Acoust. Soc. Amer., 58(2) pp. 365-370.

9. Abramowitz, Milton, and Irene Stegun, *Handbook of Mathematical Functions*, National Bureau of Standards Applied Mathematics Series 55, Washington DC, 1965.

10. Hackman, Roger, and Gary Steven Sammelmann, (1986), "Acoustic Scattering in an Inhomogeneous Waveguide: Theory", J. Acoust. Soc. Amer., 80(5), pp. 1447-1458.

11. Hackman, Roger, and Gary Steven Sammelmann, (1988), "Long Range Scattering in a Deep Oceanic Waveguide", J. Acoust. Soc. Amer., 83(5), pp 1776-1793.

12. Sammelmann, Gary Steven, and Roger Hackman, (1987), "Acoustic Scattering in a Homogeneous Waveguide", J. Acoust. Soc. Amer., 82(1), pp. 324-336.

13. Tinkham,Michael, (1964), *Group Theory and Quantum Mechanics*, (Mc Graw-Hill Book Company, New York.

14. Helgason,Sigurdur, (1978), *Differential Geometry, Lie Groups, and Symmetric Spaces*, (Academic Press, New York)

15. Weyl, Hermann, (1946), *The Classical Groups: Their Invariants and Representations*, (Princeton University Press, Princeton New Jersey)

16. Waterman, P.C., (1968), "New Formulation of Acoustic Scattering", J. Acoust. Soc. Amer., 45(6), pp. 1417-1429.

17. Hackman, Roger H., "Acoustic Scattering from Elastic Solids", in *Underwater Scattering and Radiation: Physical Acoustics Volume XXII* (1993), edited by Allan D. Pierce, and R. N. Thurston, (Academic Press, New York)

18. Gradshetytn, I.S., I.M. Rhyzik, *Table of Integrals, Series, and Products*, (Academic Press, New York), 1980.

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